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Analytic and Parameter-Free Formula for the Neutrino Mixing Matrix

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Abstract

A parameter-free analytic expression for the PMNS matrix is derived which fits numerically all the measured matrix components at 99.7% confidence. Results are proven within the microscopic model and include a prediction of the leptonic Jarlskog invariant. The approach is universal in the sense that it can be applied to the quark sector as well. Preliminary numbers obtained for the CKM matrix elements look promising, but are plagued with large theoretical errors.

Introduction

As well known there is a mixing between the flavor and mass eigenstates of the 3 neutrino species, and this can be described by a unitary matrix, the PMNS neutrino mixing matrix[1, 2]. The experimentally relevant quantities are the absolute values of the matrix elements, which describe the amount of admixture of the flavor into mass eigenstates, and the leptonic Jarlskog invariant which describes any possible CP violation in the leptonic sector.

Since the discovery of neutrino oscillations, many models of neutrino mass and mixings have been constructed. The most straightforward approach is to incorporate Dirac neutrino masses into the Standard Model by introducing three right-handed neutrinos coupled to a Higgs field analogously to the quarks and charged leptons.

Unfortunately, within the SM the values of the mixing parameters cannot be predicted.

Leading symmetric Approximation

In a first step a leading order result for the mixing matrix will be derived which is

$$\begin{aligned} V_{PMNS} &= \exp\left\{ \frac{i}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \right\} \\ &= \begin{bmatrix} 0.8467 - i0.0300 & -0.1489 + i0.4861 & 0.1532 - i0.00051 \\ -0.1489 - i0.4861 & 0.5446 + i0.4568 & -0.00433 - i0.4858 \\ 0.1532 - i0.00051 & -0.00433 - i0.4858 & 0.6892 - i0.5153 \end{bmatrix} \quad (1) \end{aligned}$$

while an improved formula will be given later in (28).

The leading order expression (1) is a complex, symmetric and unitary matrix, and the absolute values of the matrix elements can be calculated numerically and compared to measurements

$$\begin{bmatrix} 0.843 & 0.510 & 0.153 \\ 0.510 & 0.711 & 0.486 \\ 0.153 & 0.486 & 0.861 \end{bmatrix} \quad vs. \quad \begin{bmatrix} 0.80 - 0.85 & 0.51 - 0.58 & 0.142 - 0.155 \\ 0.23 - 0.51 & 0.46 - 0.69 & 0.63 - 0.78 \\ 0.25 - 0.53 & 0.47 - 0.70 & 0.61 - 0.76 \end{bmatrix} \quad (2)$$

By inspection one concludes that the agreement is reasonable but not optimal, with the 23 entry being the most critical. The first row, which is best measured, is

also best fitting. Concerning the other rows, the experimental results in (2) are non-symmetric, though with very large errors. It will be described later, in connection with (28) and (29), how (1) can be improved by additional non-symmetric contributions so that complete agreement within the errors is obtained.

A prediction for the leptonic Jarlskog invariant[3] can be calculated from (1) as

$$J_{PMNS} = \Im(V_{e1} V_{\mu 2} \bar{V}_{e2} \bar{V}_{\mu 1}) = -0.0106 \tag{3}$$

This value is large as compared to the Jarlskog parameter of the CKM matrix[4]. J_{PMNS} has not been measured so far, although there are experimental indications that leptonic CP violation is indeed rather large[5].

Motivation and Proof

The model, on which the proof is based[6, 7], starts from a fundamental isospin doublet field $\Psi = (\psi_\uparrow, \psi_\downarrow)$ consisting of two SO(3,1) Dirac fields $\psi_\uparrow$ and $\psi_\downarrow$. Ordinary matter quarks and leptons are considered as excitations of isospin vectors

$$\vec{Q}_L = \frac{1}{4}\Psi^\dagger(1-\gamma_5)\vec{\tau}\Psi \qquad\qquad \vec{Q}_R = \frac{1}{4}\Psi^\dagger(1+\gamma_5)\vec{\tau}\Psi \tag{4}$$

of the Ψ-field, namely as fluctuations $\delta\vec{Q}_L$ and $\delta\vec{Q}_R$ of the ground state values $\langle\vec{Q}_L\rangle$ and $\langle\vec{Q}_R\rangle$. $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ are the Pauli matrices in 'internal' isospin space, whose coordinates will be denoted as x, y and z.

Note that the corresponding excitations $\delta\Psi$ are fermions, but their dynamics can best be described in terms of isospin vectors (4). Namely, mass eigenvalues can be calculated using Hamiltonians H which involve interactions of the isospin vectors and then diagonalizing the equations

$$\frac{d\vec{Q}_{L,R}}{dt} = i\,[H, \vec{Q}_{L,R}] \tag{5}$$

Assuming a suitable tetrahedral configuration for the isospin vectors, 24 eigenvalues arise from (5), which are interpreted as the quark and lepton masses[6, 7].

While the masses correspond to the eigenvalues, CKM and PMNS mixings can be deduced from the eigenvectors. The relation between the eigenvectors, the mass eigenstates and the weak interaction eigenstates are clarified in the following discussion. Thereby, the result (1) and its improvement (28) for the PMNS matrix will be obtained.

The first step is to label the quark and lepton mass states in terms of the vectors $\delta\vec{Q}$. More in detail, the following definitions are used:

$$|\vec{S}\rangle = \delta\vec{Q}_L \qquad\qquad |\vec{T}\rangle = \delta\vec{Q}_R \tag{6}$$

Dirac's notation with bra and ket states is applied here to make the mixing relations more transparent. In fact, (6) are orthonormal vector states and can be used to write down the equations for the neutrino mass eigenstates, as obtained from the diagonalization procedure[7]

$$\begin{aligned}
|\nu_{e,m}\rangle &= \frac{1}{\sqrt{6}}[(|S_x\rangle + |T_x\rangle) + (|S_y\rangle + |T_y\rangle) + (|S_z\rangle + |T_z\rangle)] \\
|\nu_{\mu,m}\rangle &= \frac{1}{\sqrt{6}}[(|S_x\rangle + |T_x\rangle) + \omega(|S_y\rangle + |T_y\rangle) + \bar{\omega}(|S_z\rangle + |T_z\rangle)] \\
|\nu_{\tau,m}\rangle &= \frac{1}{\sqrt{6}}[(|S_x\rangle + |T_x\rangle) + \bar{\omega}(|S_y\rangle + |T_y\rangle) + \omega(|S_z\rangle + |T_z\rangle)]
\end{aligned} \tag{7}$$

The corresponding result for the charged leptons is

$$\begin{aligned}
|e_m\rangle &= \frac{1}{\sqrt{6}}[(|T_x\rangle - |S_x\rangle) + (|T_y\rangle - |S_y\rangle) + (|T_z\rangle - |S_z\rangle)] \\
|\mu_m\rangle &= \frac{1}{\sqrt{6}}[(|T_x\rangle - |S_x\rangle) + \omega(|T_y\rangle - |S_y\rangle) + \bar{\omega}(|T_z\rangle - |S_z\rangle)] \\
|\tau_m\rangle &= \frac{1}{\sqrt{6}}[(|T_x\rangle - |S_x\rangle) + \bar{\omega}(|T_y\rangle - |S_y\rangle) + \omega(|T_z\rangle - |S_z\rangle)]
\end{aligned} \tag{8}$$

The appearance of the complex numbers

$$\omega = -\frac{1 - i\sqrt{3}}{2} \qquad\qquad \bar{\omega} = -\frac{1 + i\sqrt{3}}{2} \tag{9}$$

corresponding to rotations by 120 and 240 degrees are an effect of the underlying tetrahedral symmetry. They turn the expressions (7) and (8) into symmetry adapted functions.

The lepton mass states actually can be brought to the much more compact form

$$\begin{bmatrix} |\nu_{em}\rangle \\ |\nu_{\mu m}\rangle \\ |\nu_{\tau m}\rangle \end{bmatrix} = Z \begin{bmatrix} |V_x\rangle \\ |V_y\rangle \\ |V_z\rangle \end{bmatrix} \qquad\qquad \begin{bmatrix} |e_m\rangle \\ |\mu_m\rangle \\ |\tau_m\rangle \end{bmatrix} = Z \begin{bmatrix} |A_x\rangle \\ |A_y\rangle \\ |A_z\rangle \end{bmatrix} \tag{10}$$

by using the quantities

$$|\vec{V}\rangle = \frac{1}{\sqrt{2}}(|\vec{S}\rangle + |\vec{T}\rangle) \qquad\qquad |\vec{A}\rangle = \frac{1}{\sqrt{2}}(|\vec{T}\rangle - |\vec{S}\rangle) \tag{11}$$

and the Z_3 Fourier transform matrices

$$Z = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{bmatrix} \qquad Z^\dagger = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{\omega} & \omega \\ 1 & \omega & \bar{\omega} \end{bmatrix} \tag{12}$$

It is interesting to note that the eigenfunctions (7), (8) and (10) are stable against variations of all the isospin couplings one may use in the Hamiltonian H in (5). In consequence, the neutrino mixing matrix does not depend on any fermion mass values. This implies a stable and unambiguous prediction for the PMNS matrix and is in contrast to the CKM matrix in the quark sector, where a mass dependence shows up, cf. Eq. (39) later.

As well known, the defining equation for the PMNS matrix is

$$\begin{bmatrix} \langle \nu_{ew}| & \langle \nu_{\mu w}| & \langle \nu_{\tau w}| \end{bmatrix} W_\mu^+ \begin{bmatrix} |e_w\rangle \\ |\mu_w\rangle \\ |\tau_w\rangle \end{bmatrix} = \begin{bmatrix} \langle \nu_{em}| & \langle \nu_{\mu m}| & \langle \nu_{\tau m}| \end{bmatrix} W_\mu^+ V_{PMNS} \begin{bmatrix} |e_m\rangle \\ |\mu_m\rangle \\ |\tau_m\rangle \end{bmatrix} \tag{13}$$

where the index w denotes weak interaction eigenstates, and it is understood that we talk about left handed fields only. The mixing matrix is formally given by

$$V_{PMNS} = V_N V_L^\dagger = \begin{bmatrix} V_{1e} & V_{1\mu} & V_{1\tau} \\ V_{2e} & V_{2\mu} & V_{2\tau} \\ V_{3e} & V_{3\mu} & V_{3\tau} \end{bmatrix} \tag{14}$$

where

$$V_N = \begin{bmatrix} \langle \nu_{em}| \\ \langle \nu_{\mu m}| \\ \langle \nu_{\tau m}| \end{bmatrix} \begin{bmatrix} |\nu_{ew}\rangle & |\nu_{\mu w}\rangle & |\nu_{\tau w}\rangle \end{bmatrix} \qquad V_L^\dagger = \begin{bmatrix} \langle e_w| \\ \langle \mu_w| \\ \langle \tau_w| \end{bmatrix} \begin{bmatrix} |e_m\rangle & |\mu_m\rangle & |\tau_m\rangle \end{bmatrix} \tag{15}$$

Replacing the mass eigenstates by the isospin excitations according to (10) one obtains

$$V_{PMNS} = Z \left\{ \begin{bmatrix} \langle V_x| \\ \langle V_y| \\ \langle V_z| \end{bmatrix} \begin{bmatrix} |\nu_{ew}\rangle & |\nu_{\mu w}\rangle & |\nu_{\tau w}\rangle \end{bmatrix} \begin{bmatrix} \langle e_w| \\ \langle \mu_w| \\ \langle \tau_w| \end{bmatrix} \begin{bmatrix} |A_x\rangle & |A_y\rangle & |A_z\rangle \end{bmatrix} \right\} Z^\dagger \tag{16}$$

By inspection one sees that (16) exactly compensates all the matrix transformations in (13) and (10) so as to maintain lepton universality and keep the weak current diagonal in the weak eigenstates.

The brace in (16) comprises a matrix of expectation values of the form

$$Y := \begin{bmatrix} \langle V_x| \\ \langle V_y| \\ \langle V_z| \end{bmatrix} \mathcal{O} \begin{bmatrix} |A_x\rangle & |A_y\rangle & |A_z\rangle \end{bmatrix} \tag{17}$$

where the inner product

$$\mathcal{O} := \begin{bmatrix} |\nu_{ew}\rangle & |\nu_{\mu w}\rangle & |\nu_{\tau w}\rangle \end{bmatrix} \begin{bmatrix} \langle e_w| \\ \langle \mu_w| \\ \langle \tau_w| \end{bmatrix} \tag{18}$$

is a dyadic 1-dimensional operator which acts between the complex 3-dimensional spaces of charged lepton ($\sim \vec{S} - \vec{T}$) and antineutrino ($\sim \vec{S} + \vec{T}$) states. One may say that it contains all information about what the charged W-boson does to the lepton fields: it changes isospin, mixes families and so on. Weak SU(2) and tetrahedral symmetry force $\mathcal{O}$ to have the form

$$\begin{aligned} \mathcal{O} \;=\; & |S_x\rangle\langle T_x| + |S_y\rangle\langle T_y| + |S_z\rangle\langle T_z| - |T_x\rangle\langle S_x| - |T_y\rangle\langle S_y| - |T_z\rangle\langle S_z| \\ & + \frac{i}{\sqrt{3}}[|S_y\rangle\langle S_z| + |S_z\rangle\langle S_y| - |T_y\rangle\langle T_z| - |T_z\rangle\langle T_y|] \\ & + \frac{i}{\sqrt{3}}[\omega\,|S_x\rangle\langle S_y| + \bar{\omega}\,|S_y\rangle\langle S_x| - \omega\,|T_x\rangle\langle T_y| - \bar{\omega}\,|T_y\rangle\langle T_x|] \\ & + \frac{i}{\sqrt{3}}[\bar{\omega}\,|S_x\rangle\langle S_z| + \omega\,|S_z\rangle\langle S_x| - \bar{\omega}\,|T_x\rangle\langle T_z| - \omega\,|T_z\rangle\langle T_x|] \end{aligned} \tag{19}$$

In order to derive (19) one has to note that SU(2) invariance allows the appearance of dot products and triple products only. The coefficients of these products are then dictated by the tetrahedral symmetry of the isospin vectors. For example, to derive the triple product coefficients one should remember that the W^+-boson is defined in the 3 internal dimensions in an analogous manner as a plus circularly polarized wave in 3 spatial dimensions, namely by means of an (internal) 'polarization vector' $\vec{e}_+ = (\vec{e}_1 + i\vec{e}_2)/\sqrt{2}$ which is perpendicular to the axis of quantization, in this case given by $\sim (1,1,1)$.

$$\vec{e}_1 = \frac{1}{\sqrt{2}}(0,1,-1) \qquad\qquad \vec{e}_2 = \frac{1}{\sqrt{6}}(-2,1,1) \tag{20}$$

Introducing the vector

$$\vec{\Omega} = \frac{1}{\sqrt{3}}(1, \omega, \bar{\omega}) \tag{21}$$

allowed contributions to $\mathcal{O}$ are of the triple product form

$$\varepsilon_{ijk}\frac{1}{\sqrt{2}}(\vec{e}_1 + i\vec{e}_2)_i \, |Q_j\rangle \langle Q'_k| = -\frac{i}{\sqrt{3}} \vec{\Omega}(\vec{Q} \times \vec{Q}') = -\frac{i}{\sqrt{3}} [\, |Q'_y\rangle \langle Q_z| - |Q'_z\rangle \langle Q_y| \\ -\omega(|Q'_x\rangle \langle Q_z| - |Q'_z\rangle \langle Q_x|) + \bar{\omega}(|Q'_x\rangle \langle Q_y| - |Q'_y\rangle \langle Q_x|\,)\,] \tag{22}$$

for the ket and bra states belonging to any 2 internal angular momenta Q and Q'. These contributions are anti-hermitian, and care must be taken in the definition of the complex triple product when using complex conjugation in the determination of $\mathcal{O}$.

Note that $\mathcal{O}$ as given in (19) is universal in the sense that it depends only on properties of the Ψ field, and therefore will appear in identical form within the quark sector and the calculation of the CKM matrix. This fact reflects the quark lepton universality of the W-boson interactions.

Inserting (19) into (17) one obtains

$$Y = \begin{bmatrix} \langle V_x| \\ \langle V_y| \\ \langle V_z| \end{bmatrix} \mathcal{O} \begin{bmatrix} |A_x\rangle & |A_y\rangle & |A_z\rangle \end{bmatrix} = I + X \tag{23}$$

i.e. a sum of a hermitian part (the unit matrix I) and an anti-hermitian matrix

$$X = -\frac{i}{\sqrt{3}} \begin{bmatrix} 0 & \bar{\omega} & \omega \\ \omega & 0 & 1 \\ \bar{\omega} & 1 & 0 \end{bmatrix} \tag{24}$$

The invariant structure which gives the unit matrix in (23) is the dot product, while the invariant structure belonging to the anti-hermitian contribution X is the triple product. The unit matrix corresponds to no mixing at all, so the origin of a non-trivial PMNS matrix is to be found solely in the triple product terms (22).

***(folgender Satz ist neu) The result (23) is anti-hermitian and not unitary, because it represents the leading term in the series exp(X) to enter the unitary PMNS matrix

in the following way

$$\begin{aligned} V_{PMNS} &= Ze^{X}Z^{\dagger} = e^{ZXZ^{\dagger}} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{bmatrix} \exp\left\{ \frac{-i}{\sqrt{3}} \begin{bmatrix} 0 & \bar{\omega} & \omega \\ \omega & 0 & 1 \\ \bar{\omega} & 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{\omega} & \omega \\ 1 & \omega & \bar{\omega} \end{bmatrix} \\ &= \begin{bmatrix} 0.8467 - i0.0300 & -0.1489 + i0.4861 & 0.1532 - i0.00051 \\ -0.1489 - i0.4861 & 0.5446 + i0.4568 & -0.00433 - i0.4858 \\ 0.1532 - i0.00051 & -0.00433 - i0.4858 & 0.6892 - i0.5153 \end{bmatrix} \end{aligned} \tag{25}$$

Note that this is identical to what was claimed in (1).

Improved Formula for the PMNS Matrix

So far only dot product and triple product terms (22) have been considered as contributing to the operator (19) and the PMNS result. Actually, there is a third kind of term that needs consideration. Using $\vec{\Omega}^2 = 0$ and the same normalization as in (22) it is of the form

$$-(\vec{\Omega} \times \vec{Q})\,(\vec{\Omega} \times \vec{Q}') = (\vec{\Omega}\vec{Q})\,(\vec{\Omega}\vec{Q}') \tag{26}$$

In the microscopic model, quark and lepton masses are related to torsional, Heisenberg and Dzyaloshinskii isospin interactions of the fundamental Ψ field. Furthermore, as shown in [11], these three types of interactions completely fix the structure of the model.

This fact is reflected in the contributions to the operator $\mathcal{O}$: while the dot products and triple products appearing in (19) parallel the torsional and Heisenberg interactions, (26) corresponds to the Dzyaloshinskii Hamiltonian. Working out the products $|Q_i\rangle \langle Q'_j|$ arising from (26), it leads to an additional contribution to (19) which can be comprised by a matrix

$$D := \frac{1}{3}\begin{bmatrix} 1 & \omega & \bar{\omega} \\ \omega & \bar{\omega} & 1 \\ \bar{\omega} & 1 & \omega \end{bmatrix} \tag{27}$$

The role of D for (26) is analogous to that of X for the triple product term. Combining the X and D contributions an improved formula for the PMNS matrix is

obtained

$$V_{PMNS} = \exp\left\{\frac{1}{3}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}\right\}\exp\left\{\frac{i}{\sqrt{3}}\begin{bmatrix}0 & 1 & 0\\ 1 & 1 & -1\\ 0 & -1 & -1\end{bmatrix}\right\} \tag{28}$$

This represents a complex and unitary matrix whose absolute value matrix $|V_{PMNS}|$ is not symmetric, in contrast to (1). Its elements are given by

$$\begin{bmatrix}0.847 & 0.510 & 0.153\\ 0.468 & 0.581 & 0.666\\ 0.251 & 0.635 & 0.730\end{bmatrix} \quad vs. \quad \begin{bmatrix}0.80-0.85 & 0.51-0.58 & 0.142-0.155\\ 0.23-0.51 & 0.46-0.69 & 0.63-0.78\\ 0.25-0.53 & 0.47-0.70 & 0.61-0.76\end{bmatrix} \tag{29}$$

and fit the phenomenological numbers to within one standard error.

The value of the leptonic Jarlskog invariant now is

$$J_{PMNS} = 0.0454 \tag{30}$$

Thus, while the improvement (28) only moderately corrects the absolute values, it strongly modifies the prediction for J_{PMNS}. This is because - in contrast to the absolute values - the Jarlskog invariant is dominated by higher orders of the exponential expansion.

Application to the Quark Sector

Mixing in the quark sector has been known since the time of Cabibbo[8]. Although the mixing percentages are smaller, it is much better measured than in the lepton sector. On the other hand, concerning theory, the predictions for the CKM mixing elements in the present model are somewhat more difficult to obtain, though parts of the arguments for leptons can be taken over to the quark sector. The idea is again that the mixing matrix counterbalances the deviation of the mass eigenstates from the weak eigenstates in such a way that the charged current effectively acts diagonal on the isospin operators (6). The main complication is the appearance of mass dependent factors in the quark eigenstates, see below.

The CKM matrix is defined analogously to the PMNS matrix (14) and (15)

$$\begin{aligned} V_{CKM} &= V_U V_D^\dagger = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \\ &= \begin{bmatrix} \langle u_m|u_w\rangle & \langle u_m|c_w\rangle & \langle u_m|t_w\rangle \\ \langle c_m|u_w\rangle & \langle c_m|c_w\rangle & \langle c_m|t_w\rangle \\ \langle t_m|u_w\rangle & \langle t_m|c_w\rangle & \langle t_m|t_w\rangle \end{bmatrix} \begin{bmatrix} \langle d_w|d_m\rangle & \langle d_w|s_m\rangle & \langle d_w|b_m\rangle \\ \langle s_w|d_m\rangle & \langle s_w|s_m\rangle & \langle s_w|b_m\rangle \\ \langle b_w|d_m\rangle & \langle b_w|s_m\rangle & \langle b_w|b_m\rangle \end{bmatrix} \end{aligned} \quad (31)$$

where m denotes mass eigenstates (the physical states) and w weak interaction eigenstates.

Solving the eigenvalue problem (5) leads to mass eigenstates for the up-type quarks

$$\begin{aligned} u_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_1^2}}[(|S_x\rangle + \epsilon_1 |T_x\rangle) + (|S_y\rangle + \epsilon_1 |T_y\rangle) + (|S_z\rangle + \epsilon_1 |T_z\rangle)] \\ c_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_2^2}}[(|S_x\rangle + \epsilon_2 |T_x\rangle) + \omega(|S_y\rangle + \epsilon_2 |T_y\rangle) + \bar{\omega}(|S_z\rangle + \epsilon_2 |T_z\rangle)] \\ t_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_3^2}}[(|S_x\rangle + \epsilon_3 |T_x\rangle) + \bar{\omega}(|S_y\rangle + \epsilon_3 |T_y\rangle) + \omega(|S_z\rangle + \epsilon_3 |T_z\rangle)] \end{aligned} \quad (32)$$

and for the down quarks

$$\begin{aligned} d_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_1^2}}[(|T_x\rangle - \epsilon_1 |S_x\rangle) + (|T_y\rangle - \epsilon_1 |S_y\rangle) + (|T_z\rangle - \epsilon_1 |S_z\rangle)] \\ s_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_2^2}}[(|T_x\rangle - \epsilon_2 |S_x\rangle) + \omega(|T_y\rangle - \epsilon_2 |S_y\rangle) + \bar{\omega}(|T_z\rangle - \epsilon_2 |S_z\rangle)] \\ b_m &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_3^2}}[(|T_x\rangle - \epsilon_3 |S_x\rangle) + \bar{\omega}(|T_y\rangle - \epsilon_3 |S_y\rangle) + \omega(|T_z\rangle - \epsilon_3 |S_z\rangle)] \end{aligned} \quad (33)$$

Three coefficients $\epsilon_{1,2,3}$ appear in these equations, which depend on the quark and even on the lepton masses. They can be calculated within the model. Namely, as proven in [11], each ϵ_i to a very good approximation only depends on the quark and charged lepton masses of the i-th family. More precisely, one can derive the formula[11]

$$\epsilon_i = \frac{1}{6}\frac{M_{Li}}{M_{Ui} + M_{Di}} \quad (34)$$

where M_{Ui}, M_{Di} and M_{Li} denote the corresponding masses within family i.

By inspection one sees that the lepton eigenfunctions (7) and (8) are recovered from (32) and (33) by chosing $\epsilon_3 = \epsilon_2 = \epsilon_1 = 1$. It should be stressed, however, that this is only formally true, because the quark states are defined in a different space than the lepton states. The point is that for simplicity reference has been made so far to only one of the four isospins I, II, III and IV on the tetrahedral structure. While the contributions from I-IV to the lepton states are identical and of the form I+II+III+IV, the generic form of the quark states turns out to be 3×I-II-III-IV, 3×II-I-III-IV and 3×III-II-IV for the 3 colors, respectively.

Knowing the eigenstates (32) and (33) one may write down the CKM matrix in an analogous fashion as the PMNS matrix (16) for leptons

$$V_{CKM} = \left\{ RZ \begin{bmatrix} \langle S_x| \\ \langle S_y| \\ \langle S_z| \end{bmatrix} + REZ \begin{bmatrix} \langle T_x| \\ \langle T_y| \\ \langle T_z| \end{bmatrix} \right\} \begin{bmatrix} |u_w\rangle & |c_w\rangle & |t_w\rangle \end{bmatrix} \begin{bmatrix} \langle d_w| \\ \langle s_w| \\ \langle b_w| \end{bmatrix} \times$$
$$\times \left\{ \begin{bmatrix} |T_x\rangle & |T_y\rangle & |T_z\rangle \end{bmatrix} Z^\dagger R - \begin{bmatrix} |S_x\rangle & |S_y\rangle & |S_z\rangle \end{bmatrix} Z^\dagger E R \right\} \tag{35}$$

where the matrices

$$E := \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \qquad R := \begin{bmatrix} \frac{1}{\sqrt{1+\epsilon_1^2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+\epsilon_2^2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1+\epsilon_3^2}} \end{bmatrix} \tag{36}$$

have been introduced.

Just as in the case of leptons (18) there is a 1-dimensional dyadic transformation

$$\mathcal{O} = \begin{bmatrix} |u_w\rangle & |c_w\rangle & |t_w\rangle \end{bmatrix} \begin{bmatrix} \langle d_w| \\ \langle s_w| \\ \langle b_w| \end{bmatrix} \tag{37}$$

which operates between the 3-dimensional spaces of up- and down-type quark states. Due to quark-lepton universality, when expressed in terms of operators $\vec{S}$ and $\vec{T}$, the operator $\mathcal{O}$ for quarks must be identical to what was used for leptons in (19).

Restricting, for a moment, on the dot and triple product contributions (19) as input, one may then calculate V_{CKM} given in (35) to be

$$V_{CKM} = I + RZXZ^\dagger ER + REZXZ^\dagger R \to \exp\{RZXZ^\dagger ER + REZXZ^\dagger R\} \tag{38}$$

where I is the 3×3 unit matrix arising from the dot product terms in (19). The other terms in (38) are the anti-hermitian contributions from the triple product in (22) and (19). They replace the expression $ZXZ^\dagger$ in (25) for leptons.

Just as in the case of leptons one may improve on this result by including the contributions from (26), in order to obtain the desired non-symmetric contributions to $|V_{CKM}|$. The improved formula for the CKM matrix reads

$$V_{CKM} = \exp\{2[RZDZ^\dagger ER - REZD^\dagger Z^\dagger R]\} \exp\{RZXZ^\dagger ER + REZXZ^\dagger R\} \quad (39)$$

In contrast to X in (24) the matrix D in (27) is not anti-hermitian. This fact has been accounted for in the first exponential factor.

Eq. (39) allows to evaluate $|V_{CKM}|$ using appropriate values for the fermion masses entering (34). It must be noted, however, that the low energy values of the ϵ_i are not useful in this context. Instead one should use running masses near the Planck scale, because the dynamics generates fermion masses originally at Planck scale distances[1]. Unfortunately, the predictions for running masses are not very precise because higher order contributions become appreciable at very large scales. Nevertheless, I am using results from the literature[9, 10] to determine the ϵ_i at high scales.

$$\epsilon_1 = 0.35 \qquad \epsilon_2 = 0.070 \qquad \epsilon_3 = 0.0040 \quad (40)$$

unfortunately with a large theoretical error[10], whose magnitude even is hard to estimate. The numbers are for a 2HDM (2 Higgs doublet model) which is known to be the low-energy limit of the microscopic model[6]. They exhibit a family hierarchy which will be seen to induce a corresponding hierarchy in the mixing of the quark families. Actually, as discussed in earlier work[7], this is to be expected within the present model due to the large top mass which forces the up- and down-type mass eigenstates to be approximately $\sim \vec{S}$ and $\sim \vec{T}$, respectively, in (32) and (33), much unlike the lepton states which are $\sim \vec{S} \pm \vec{T}$ according to (10).

Just as masses, CKM matrix elements are running, i.e. dependent on the scale paramter $t = \ln \frac{E}{\mu}$ where E is the relevant energy scale and μ the renormalization

[1] A GUT scale is not present in the model. There is only the Fermi scale, defined as the interaction energy of the isospin vectors, and the Planck scale, defined as the binding energy of the fields Ψ[6].

scale. The running of the absolute values of the CKM matrix elements has been discussed for the 2HDM in [10]. It turns out to be remarkably simple, at least in leading order, because it can be given in terms of one universal function h(t).

$$|V_{CKM}(t)| \approx \begin{bmatrix} |V_{ud}(0)| & |V_{us}(0)| & \frac{|V_{ub}(0)|}{h(t)} \\ |V_{cd}(0)| & |V_{cs}(0)| & \frac{|V_{cb}(0)|}{h(t)} \\ \frac{|V_{td}(0)|}{h(t)} & \frac{|V_{ts}(0)|}{h(t)} & |V_{tb}|(0) \end{bmatrix} \tag{41}$$

For the Jarlskog invariant one has

$$J_{CKM}(t) \approx \frac{J_{CKM}(0)}{h^2(t)} \tag{42}$$

In the 2HDM case $h(t)$ is a moderately varying function. According to [10] it increases by about 20% when going from GeV to Planck scale energies.

Using (39) and (40) I have calculated the CKM elements at high energies and then extrapolated them back to GeV energies according to (41). I obtain the matrix $|V_{CKM}|$ of absolute values

$$\begin{bmatrix} 0.974 & 0.224 & 0.0035 \\ 0.224 & 0.973 & 0.044 \\ 0.0080 & 0.043 & 0.9991 \end{bmatrix} vs. \begin{bmatrix} 0.9734 - 0.9740 & 0.2235 - 0.2251 & 0.00362 - 0.00402 \\ 0.217 - 0.225 & 0.969 - 0.981 & 0.0394 - 0.0422 \\ 0.0083 - 0.0088 & 0.0404 - 0.0424 & 0.985 - 1.043 \end{bmatrix} \tag{43}$$

The numbers look reasonable, as compared to the phenomenological values, and show the correct hierarchy and orders of magnitude. However, the theoretical uncertainty from the scale evolution is large and difficult to estimate, in particular concerning quark mass values near the Planck scale. For example, ϵ_1 accommodates the Cabbibo angle correctly, whereas the '23'-matrix elements $|V_{ts}|$ and $|V_{cb}|$ tendencially come out too large, while the '13'-elements $|V_{ub}|$ and $|V_{td}|$ are typically too small. These deviations may seem being just 2σ effects, but as stressed before the theoretical error from the quark mass evolution is extremely difficult to handle.

Similarly, concerning the Jarlskog invariant one obtains $J_{CKM} = 0.000027$, a bit small when compared to the observed value $J_{CKM} = (3.00 + 0.15 - 0.09) \times 10^{-5}$.

In conclusion, explicit analytic and numerical results for the mixing matrices have been presented in this work. Of particular interest are the prediction for the PMNS matrix (28) and the fermion mass dependence of the CKM matrix as given by (39).

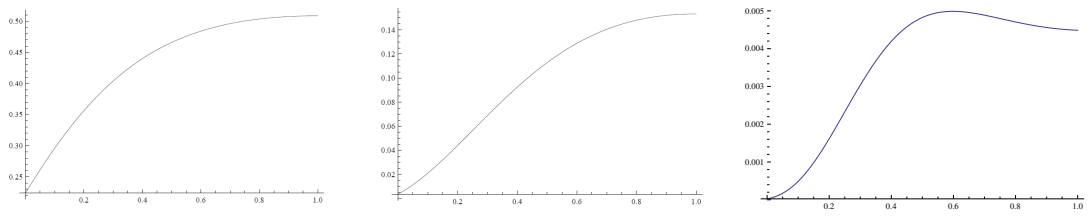

Figure 1: Transition between the CKM and the PMNS limit of the matrix elements $|V_{12}|$ and $|V_{13}|$ and the Jarlskog invariant (from left to right) as a function of the parameter α defined in the main text. For example, $|V_{12}|$ starts with its CKM value 0.224 at $\alpha = 0$ and grows towards the PMNS value at $\alpha = 1$.

Actually, (39) is universal in that it embraces (i) the case of no mixing ($\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$), (ii) the CKM prediction obtained with ϵ_i-values (40) and (iii) the PMNS formula which formally is given using $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. To make this visible, I have drawn in Fig. 1 the '12' (i.e. Cabibbo) and the '13' matrix element and the Jarlskog invariant as a function of a parameter α. α is introduced to avoid drawing the full ϵ_i-dependence of the matrix elements and defined in such a way that it vanishes in the CKM case and takes the value of 1 in the PMNS limit. More precisely, one has

$$\epsilon_1 = 0.35 + 0.65\,\alpha \qquad \epsilon_2 = 0.07 + 0.93\,\alpha \qquad \epsilon_3 = 0.004 + 0.996\,\alpha \tag{44}$$

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