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World of Tetrons

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Abstract

A model is presented, in which fermion and vector boson states are constructed from constituents ('tetrons'). The model encodes all observed structures and phenomena of elementary particle physics in group theoretic items of the permutation group S_4. Details of the model like symmetry breaking, distribution of charges and mass generation are worked out.

1 Introduction

According to present ideas the observed elementary particles (leptons, quarks and vector bosons) are pointlike. Their mathematical description [1] as Dirac or Yang-Mills fields follows this philosophy. In the present paper I propose a model, in which they acquire an extension and are composite of more fundamental fields called tetrons.

2 The Model for Fermions

My aim is to shed light on the fermion spectrum of elementary particle physics, i.e. the 24 spin-$\frac{1}{2}$-states observed in nature and habitually denoted by

$$\nu_e \qquad e \qquad u_{1,2,3} \qquad d_{1,2,3} \tag{1}$$

$$\nu_\mu \qquad \mu \qquad c_{1,2,3} \qquad s_{1,2,3} \tag{2}$$

$$\nu_\tau \qquad \tau \qquad t_{1,2,3} \qquad b_{1,2,3} \tag{3}$$

where the index 1,2,3 stands for quark color. These states arrange themselves in 3 families of

$$N \qquad E \qquad U_{1,2,3} \qquad D_{1,2,3} \tag{4}$$

each consisting of two quadruplets of the form $(L, Q_{1,2,3})$ where I abbreviate quark states by the letter Q, up-quarks by U, down-quarks by D, leptons by L, neutrinos by N and e, μ and τ by E. Including spin (and righthanded neutrinos) there are altogether 48 degrees of freedom with a mass spectrum which ranges between about 10^{-6} eV and 175 GeV [2] [3].

The underlying dynamics of this system is nowadays usually supposed to be a local gauge theory with gauge group $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$

[4], $SU(4) \times SU(2)_L \times SU(2)_R$ [5] or $SO(10)$ [6] with 15, 21 or 45 gauge bosons respectively plus an as yet unknown mechanism which takes care of the family repetition.

I want to accomodate this system as bound states of smaller, more fundamental objects. The neatest idea [8] is to consider states consisting of 4 identical particles ('tetrons') bound together by a new super strong, super short range interaction, whose charge and symmetry nature are not discussed at this point but will become transparent in the course of discussion. The tetrons are sitting on the corner of a tetrahedron which transforms under the permutation group S_4 [7] thus yielding the observed spectrum of quarks and leptons.

Looked at in the cms the 4-tetron-state arranges itself automatically as a completely symmetric equilateral tetrahedron with corresponding symmetry properties, i.e. invariance under the tetrahedron's symmetry group T_d which is isomorphic to the group S_4 of permutations of the set of the 4 tetrons which for convenience we numerate as 1,2,3,4. The group consists of 24 permutations which may be denoted as $\sigma = \overline{abcd} : (1, 2, 3, 4) \rightarrow (a, b, c, d)$. In the limit of perfect S_4-symmetry the 24 tetrahedron states $\phi_\sigma = |abcd>$ generated by these permutations are of course all identical with identical masses and charges.

In order to become different with different masses and charges there must be some sort of symmetry breaking, the simplest possibility being that the 4 originally identical tetrons differ from each other. In order to implement this there are several possibilities which will be described later. At the moment I simply assume that a symmetry breaking exists and want to show how the 24 different permutation states can be arranged in parallel to the observed family structure.

The nonabelian group S_4 may be written as a semidirect product

$$S_4 = Z_4 \diamond Z_3 \diamond Z_2 \tag{5}$$

where Z_n is the (abelian) symmetric group of n elements. The subgroup $Z_3 \diamond Z_2$ of S_4 can be identified with the symmetric group S_3 of 3 elements, whereas $Z_4 \diamond Z_2$ is isomorphic to the dihedral group D_4. [1]

The Z_3-part of the decomposition (5) allows to divide the 24 elements of S_4 into 3 groups of 8 elements ('orbits'), which correspond to permutations which preserve a certain ordering (...1234123...+backward, ...2134...+backward and ...4231...+backward) and will essentially make up the 3 fermion families.

There are then two possibilities to associate the up-half $(N, U_{1,2,3})$ of a family to elements of S_4. One is based on the so-called $Kleinsche\ Vierergruppe$ K (also a subgroup of S_4). [2] This however leads to leptoquarks (cf. the section about vector bosons), i.e. to unwanted lepton number violating processes like proton decay etc. The other possibility is to take the Z_4 in eq. (5), i.e. to identify the up-half of the first family (where by $first$ I do not necessarily mean lightest, see later) with the 3 permutations $\overline{2341}$, $\overline{3412}$, $\overline{4123}$ plus the identity. The down-half is then associated to the 'backward running' permutations $\overline{4321}$, $\overline{3214}$, $\overline{2143}$ and $\overline{1432}$.

[1] All these groups have geometrical interpretations as symmetry groups of simple geometrical objects. For example, D_4 is the symmetry group of a square, S_3 of an equilateral triangle and K that of a rectangle (everything in 2 dimensions). S_4 itself is isomorphic to the symmetry group of a unilateral tetrahedron in 3 dimensions and as such may be considered as a subgroup of SO(3). The latter fact, however, will not be relevant (because we are going to completely break S_4), until the point when parity violation of the weak interactions is discussed in the section on vector bosons.

[2] K is the smallest noncyclic group and isomorphic to $Z_2 \diamond Z_2$. Considered as a subgroup of S_4 it consists of the 3 permutations $\overline{2143}$, $\overline{3412}$, $\overline{4321}$, where 2 pairs of numbers are interchanged, plus the identity.

In table 1 there is a preliminary assignment of particle states to permutations of S_4. It should be noted, however, that this is only mnemonic, because it will later turn out that one has to use linear combinations of S_4-states.

Another way of representing the above classification is by saying that 24 elements of S_4 are divided in 6 classes which transform under its subgroup S_3. In fact, S_3 consists of 3 even permutations denoted by $id = \overline{I,II,III}$, $g_1 = \overline{III,I,II}$ and $g_2 = \overline{II,III,I}$ and 3 odd permutations denoted by $u_0 = \overline{III,II,I}$, $u_1 = \overline{II,I,III}$ and $u_2 = \overline{I,III,II}$. It can be further decomposed as $S_3 \approx Z_3 \diamond S_2$ where $S_2 = Z_2$ and Z_3=$\{id, g_1, g_2\}$ is the cyclic group of 3 elements and in geometrical terms corresponds to rotations by $\pm 2\pi/3$. This part of S_3 is assumed to account for the 3 observed fermion families, i.e. to be the family group while the S_2-part will correspond to weak isospin, i.e. flip neutrino and electron.

The down-quadruplet $(E, D_{1,2,3})$ can then be represented by a single even element of S_3 and $(N, U_{1,2,3})$ by a single odd element, while the 4 states within a quadruplet correspond to the orbits of Z_4 in S_4. The situation is summarized in the following list:

- 1.family: path ...1234... corresponding to $id = \overline{I,II,III}$ (+backward $p_2 = \overline{III,II,I}$)
- 2.family: path ...2134... corresponding to $p_1 = \overline{II,I,III}$ (+backward $g_1 = \overline{III,I,II}$)
- 3.family path ...4231... corresponding to $g_2 = \overline{II,III,I}$ (+backward $p_3 = \overline{I,III,II}$)

In geometrical terms the 3 families can be visualized as the 3 closed paths which can be drawn in a tetrahedron. The interactions among the tetrons in

	...1234... family 1	...1423... family 2	...1243... family 3
	τ, $b_{1,2,3}$	μ, $s_{1,2,3}$	e, $d_{1,2,3}$
F	$\overline{1234}(id)$	$\overline{2314}$	$\overline{3124}$
F	$\overline{2341}(z_1)$	$\overline{3142}$	$\overline{1243}(3 \leftrightarrow 4)$
F	$\overline{3412}(k_2 = z_2)$	$\overline{1423}$	$\overline{2431}$
F	$\overline{4123}(z_3)$	$\overline{4231}(1 \leftrightarrow 4)$	$\overline{4312}$
	ν_τ, $t_{1,2,3}$	ν_μ, $c_{1,2,3}$	ν_e, $u_{1,2,3}$
B	$\overline{3214}(1 \leftrightarrow 3)$	$\overline{1324}(2 \leftrightarrow 3)$	$\overline{2134}(1 \leftrightarrow 2)$
B	$\overline{4321}(k_3)$	$\overline{4132}$	$\overline{4213}$
B	$\overline{1432}(2 \leftrightarrow 4)$	$\overline{2413}$	$\overline{3421}$
B	$\overline{2143}(k_1)$	$\overline{3241}$	$\overline{1342}$

Table 1: Preliminary list of elements of S_4 ordered in 3 families. F are the forward oriented elements (corresponding to the Z_4 subgroup of S_4), and B the backward oriented ones. Together they give D_4. z_i and k_i denote the elements of Z_4 and K, respectively, and $(a \leftrightarrow b)$ a simple permutation where a and b are interchanged. The permutations with a 4 at the last position form a S_3 subgroup of S_4 which at this point may be thought of giving the set of lepton states. Note however, that the assignment of particle states is only mnemonic, because it will turn out that one has to use linear combinations of S_4-states. Furthermore, there is still an arbitrariness as to whether the 'first' family is the lightest or the heaviest, i.e. whether the identity permutation $\overline{1234}$ corresponds to the top quark or to the electron and so on. This is simply due to the fact that I have not yet discussed the question of symmetry breaking, masses and charges, but will do so in section 4.

the i-th family runs along the path i, i=I,II or III. A given state $\phi_\sigma = |abcd>$ has thus bindings along the (open) path $a \rightarrow b \rightarrow c \rightarrow d$.

All observed Fermions have Spin $\frac{1}{2}$ and they have associated antifermions. These features are indispensable for any theory of elementary particles. How can they be accommodated in the present model? Concerning these questions I refer to a forthcoming paper [9] where it is argued that

- Assuming tetrons are given by complex scalar fields $\chi(x)$, anti-tetrons are described by $\chi^*(x)$. Antifermions arise from an 'anti-tetrahedron' configuration, where the anti-tetrahedron consists of anti-tetrons and is the tetrahedron which completes a given tetrahedron to a cube. It is this cube, by the way, and its centre and axes, which will later be used to describe vector boson states (see section 5). The 'anti-tetrahedron' is obtained by the original one by the parity inversion $P : \vec{x} \rightarrow -\vec{x}$ plus charge conjugation $C : \phi \rightarrow \phi^*$. Note that although most of the time I consider tetrahedrons at rest, the present framework will be suitable general to be applicable to relativistic tetrahedron states.

- So far we have 24 tetrahedron states ordered according to S_4 permutations. To include spin $\pm\frac{1}{2}$ one should go to the covering group $\tilde{S}_4$ of S_4, which is obtained by adding a negative identity J for 2π rotations and is a subgroup of SU_2, the double cover of SO_3. $\tilde{S}_4$ has precisely the 48 elements needed to get all fermion spin states. The idea behind this is that in the limit in which it shrinks to a point the tetrahedron as a whole acquires spin-$\frac{1}{2}$ transformation properties [9] so that each state ϕ_σ actually appears in a spin-up and a spin-down form $\phi_\sigma^\uparrow$ and $\phi_\sigma^\downarrow$. The content of the following section holds separately for the spin-up and spin-down tetrahedrons, so that for the time being I shall not differentiate. I shall come back to spin and the related question of parity violation in section 5 and in [9].

3 Alignment of States after Symmetry Breaking

As stated, to first approximation all 24 tetrahedron states are identical (symmetric limit). Although the symmetry breaking that makes them different can have various origins, it must have to do with a breaking of the new, superstrong interaction that keeps the 4 tetrons together.

In more concrete terms one may ask, what properties the tetrons need to form the 24 states and in particular to make them all different. I have considered various possibilities but will present here only the most appealing: namely, assuming non-identical tetrons, I demand that these appear in 4 different 'charge states' called $\chi(q_i, x)$, $i = 1, 2, 3, 4$ fulfilling the following selection rules: In a tetrahedron bound state with 4 tetrons

- (A) each charge q_i which can take one of 4 possible values appears once and only once
- (B) the sum of charges vanishes, i.e. the bound states are singlets under the new superstrong interaction.

From (A) we directly get 24 product states

$$\phi_\sigma(x_1, x_2, x_3, x_4) = \chi(q_{\sigma(1)}, x_1)\chi(q_{\sigma(2)}, x_2)\chi(q_{\sigma(3)}, x_3)\chi(q_{\sigma(4)}, x_4) \qquad (6)$$

corresponding to the 24 permutations $\sigma \in S_4$ and where I have used the notation $\sigma : (1, 2, 3, 4) \to (\sigma(1), \sigma(2), \sigma(3), \sigma(4))$. Problem: At this point the

24 states can still be transformed into each other by a suitable rotation of the tetrahedron (permutation of the x_i). In order to get them different I demand additionally

- (C) the existence of cores or "nuclei" with 4 different charges Q_i in the centre of each tetron i, i.e. in the corners of the tetrahedron, which are surrounded by the charges $q_{\sigma(i)}$. The range of possible values of the Q_i may be chosen identical to that of the q_i.

The wave function for the tetrahedrons then reads

$$\phi_\sigma(x_1, x_2, x_3, x_4) = \chi(q_{\sigma(1)}, Q_1, x_1)\chi(q_{\sigma(2)}, Q_2, x_2)\chi(q_{\sigma(3)}, Q_3, x_3)\chi(q_{\sigma(4)}, Q_4, x_4) \tag{7}$$

and, modulo rotations, there are now 24 different tetrahedron states indexed by $\sigma \in S_4$. Equivalently one may say, that a tetron is described by 2 quantum numbers q and Q fulfilling the above restrictions and selection rules.

Note that I have not yet specified the nature of these quantum numbers. At the moment they are just properties which distinguish the tetrons.

Using $\sigma(1) = a$, $\sigma(1) = b$, $\sigma(1) = c$ and $\sigma(1) = d$ one may rewrite the last equation in various forms:

$$\phi_\sigma = |abcd> = \phi_{\overline{abcd}} = \chi(q_a, Q_1) \otimes \chi(q_b, Q_2) \otimes \chi(q_c, Q_3) \otimes \chi(q_d, Q_4) \tag{8}$$

where I have used the notation $\sigma = \overline{abcd} : 1234 \to \overline{abcd}$.

Note that even though most of the time I consider tetrahedrons at rest, the framework eq. (8) is suitable general to be applicable to relativistic tetrahedron states by considering spacetime instead of space coordinates.

In some sense the tetrahedron is similar to a chemical molecule with nuclei Q_i and wave function clouds q_j. Conditions (A) and (B) ensure that the lowest lying orbitals cannot be infinitely filled.

To understand the breaking more clearly, consider the simplified case of 2 clouds q_+ and q_- surrounding 2 cores Q_+ and Q_- and forming 2-core-2-cloud bound states, namely

$$\phi_{+-} = \chi(q_+, Q_+) \otimes \chi(q_-, Q_-) \tag{9}$$

and

$$\phi_{-+} = \chi(q_-, Q_+) \otimes \chi(q_+, Q_-) \tag{10}$$

and no others (i.e. assuming a modified selection rule that only bound states with 2 different clouds and 2 different cores exist). For state ϕ_{+-} cloud q_+ is nearer to core Q_+ whereas in ϕ_{-+} it is nearer to q_-, and analogously for cloud q_-. Because of all charges being different (i.e. due to the breaking of permutation symmetry) the states ϕ_{+-} and ϕ_{-+} will not be degenerate. One of them (representing the neutrino) will be lower in mass than the other (representing the electron). In nonrelativistic perturbation theory one would generically have mass formulas

$$\begin{aligned} E_{+-} &= E(q_+, Q_+) + E(q_-, Q_-) \\ &+ < \phi_{+-}|V(q_+, q_-) + V(Q_+, Q_-) + V(q_+, Q_-) + V(q_-, Q_+)|\phi_{+-} > \end{aligned} \tag{11}$$

$$\begin{aligned} E_{-+} &= E(q_-, Q_+) + E((q_+, Q_-) \\ &+ < \phi_{-+}|V(q_+, q_-) + V(Q_+, Q_-) + V(q_-, Q_-) + V(q_+, Q_+)|\phi_{-+} > \end{aligned} \tag{12}$$

where $V(a, b)$ denotes the interaction between charge a and b and $E(a, b)$ denotes the lowest order energy eigenvalues of completely separated tetrons. Mass generation by breaking terms of the new interactions will be further addressed in the section 4.

Coming back to tetrahedron configurations it must be realized that the physical states are linear combinations of the product states eq. (7). Consider, for example, the 4 states generated by applying the subgroup Z_4 on the unit element of S_4, i.e. $|1234>$, $|2341>$, $|3412>$ and $|4123>$. They are naturally built into a singlet

$$\phi_{\nu_\tau} = \frac{1}{\sqrt{4}}[\phi_{\overline{1234}} + \phi_{\overline{2341}} + \phi_{\overline{3412}} + \phi_{\overline{4123}}] \tag{13}$$

representing the τ-neutrino and 3 nonsinglet combinations

$$\phi_{t_1} = \frac{1}{\sqrt{4}}[\phi_{\overline{1234}} + i\phi_{\overline{2341}} - \phi_{\overline{3412}} - i\phi_{\overline{4123}}] \tag{14}$$

$$\phi_{t_2} = \frac{1}{\sqrt{4}}[\phi_{\overline{1234}} - \phi_{\overline{2341}} + \phi_{\overline{3412}} - \phi_{\overline{4123}}] \quad (15)$$

$$\phi_{t_3} = \frac{1}{\sqrt{4}}[\phi_{\overline{1234}} - i\phi_{\overline{2341}} - \phi_{\overline{3412}} + i\phi_{\overline{4123}}] \quad (16)$$

which are degenerate in energy, as shown in the next section, and will be used to represent the 3 color states of the top-quark.

The reason for considering these linear combinations instead of the simple product state eq. (7) is that they turn out to be eigenfunctions of the $U(1)_{B-L} \times SU(3)_c$ charge operators λ_3, λ_8 and Y_{B-L}.

To prove this, one first shows that the states (13) and (16) are eigenfunctions of permutation operators $R_0 = \overline{1234}$, $R_1 = \overline{2341}$, $R_2 = \overline{3412}$ and $R_3 = \overline{4321}$ and afterwards constructs the $U(1)_{B-L} \times SU(3)_c$ charges as linear combinations of the R_i. In fact, writing $\phi_{\nu_\tau} = (1,0,0,0)$, $\phi_{t_1} = (0,1,0,0)$ etc, the action of the R_j on the states (13) and (16) is given by $R_0 = (1,1,1,1)$, $R_1 = diag(1,-i,-1,i)$, $R_2 = diag(1,-1,1,-1)$ and $R_3 = diag(1,i,-1,-i)$. Therefore, one has $Y_{B-L} = -\frac{1}{6}(3,-1,-1,-1) = -\frac{1}{6}(R_1 + R_2 + R_3)$ and similarly for λ_3 and λ_8.

Of course, at this point Z_4 is merely a discrete symmetry. From its Casimirs λ_3, λ_8 and Y_{B-L} it should be completed to a global $U(1)_{B-L} \times SU(3)_c$ and afterwards in the limit when the tetrahedron shrinks to a point-like fermion an effective local gauge symmetry should be constructed.

Note that with the help of suitable permutations the above analysis can be extended to any other quark-lepton quadruplet. Note further that using $SU(4)$ instead of $U(1)_{B-L} \times SU(3)_c$ one would in general get leptoquarks besides gluon interactions and the $U(1)_{B-L}$-photon. These, however, can be shown to be forbidden in the present model. I shall come back to this point in section 5.

Having dealt with Z_4, one can treat the rest of $S_4 = Z_4 \diamond Z_3 \diamond Z_2$ in a similar fashion. Namely, for $S_2 = Z_2$ one may define 2 states

$$\phi_\pm = \frac{1}{\sqrt{2}}[\phi_{\overline{13}} \pm \phi_{\overline{31}}] \tag{17}$$

corresponding to eqs. (9) and (10) which are eigenstates of the generator $\overline{31}$ of S_2 with eigenvalue +1 and -1, i.e. these states should be identified with the two partners of a weak isospin doublet (like electron and its neutrino). [3]

One can easily construct a set of Pauli matrices σ_i, $i = 1, 2, 3$ for the states eq. (17) by using $\overline{31}$ as σ_3 and then defining creation and annihilation operators $\sigma_+\phi_- = \phi_+$ and $\sigma_-\phi_+ = \phi_-$ and from these $\sigma_1 = \sigma_+ + \sigma_-$ and $\sigma_2 = i(\sigma_+ - \sigma_-)$. This set of matrices is easily seen to obey SU(2) commutation relations and is going to generate the weak $SU(2)$-symmetry.

Combining eqs. (16) and (17) the real lepton and quark states get 4 additional (±) terms: neutrinos and up-type quarks with a positive, electrons and down-type quarks with a negative sign: [4]

$$\begin{aligned}
\phi_{N/E} &= \frac{1}{\sqrt{8}}[(\phi_{\overline{1234}} \pm \phi_{\overline{3214}}) + (\phi_{\overline{2341}} \pm \phi_{\overline{4321}}) \\
&\quad +(\phi_{\overline{3412}} \pm \phi_{\overline{1432}}) + (\phi_{\overline{4123}} \pm \phi_{\overline{2143}})] && (18) \\
\phi_{(U/D)_1} &= \frac{1}{\sqrt{8}}[(\phi_{\overline{1234}} \pm \phi_{\overline{3214}}) + i(\phi_{\overline{2341}} \mp \phi_{\overline{4321}}) \\
&\quad -(\phi_{\overline{3412}} \pm \phi_{\overline{1432}}) - i(\phi_{\overline{4123}} \mp \phi_{\overline{2143}})] && (19) \\
\phi_{(U/D)_2} &= \frac{1}{\sqrt{8}}[(\phi_{\overline{1234}} \pm \phi_{\overline{3214}}) - (\phi_{\overline{2341}} \pm \phi_{\overline{4321}}) \\
&\quad +(\phi_{\overline{3412}} \pm \phi_{\overline{1432}}) - (\phi_{\overline{4123}} \pm \phi_{\overline{2143}})] && (20) \\
\phi_{(U/D)_3} &= \frac{1}{\sqrt{8}}[(\phi_{\overline{1234}} \pm \phi_{\overline{3214}}) - i(\phi_{\overline{2341}} \mp \phi_{\overline{4321}})
\end{aligned}$$

[3] Note that I consider S_2 to be the permutations of two objects called here 1 and 3 to be in agreement with table 1.

[4] In the language of molecular physics these are the symmetry adapted wave function of the dihedral group D_4. The appearance of ∓ in combination with ± in eqs. (19)-(22) is the main difference between $D_4 = Z_4 \diamond Z_2$ and the direct product $Z_4 \times Z_2$.

$$-(\phi_{\overline{3412}} \pm \phi_{\overline{1432}}) + i(\phi_{\overline{4123}} \mp \phi_{\overline{2143}})] \tag{21}$$

$$\tag{22}$$

Finally we have a state-mixing due to the family group $Z_3 \subset S_3$ which means that instead of ...1234123...+backward, ...2134...+backward and ...4231...+backward of section 2 the 3 families correspond to

$$\phi_A = \frac{1}{\sqrt{3}}[\phi_{id} + \phi_{g_1} + \phi_{g_2}] \tag{23}$$

$$\phi_B = \frac{1}{\sqrt{3}}[\phi_{id} + \alpha\phi_{g_1} + \alpha^2\phi_{g_2}] \tag{24}$$

$$\phi_C = \frac{1}{\sqrt{3}}[\phi_{id} + \alpha^2\phi_{g_1} + \alpha\phi_{g_2}] \tag{25}$$

where $\alpha = exp(2i\pi/3)$ and id, g_1 and g_2 denote the elements of Z_3, i.e. even permutations of S_3, $id = \overline{123}$ $g_1 = \overline{231}$ and $g_1 = \overline{312}$. In the notation of eq. (7) one has $\phi_\sigma = \chi(q_{\sigma(1)}, Q_1) \otimes \chi(q_{\sigma(2)}, Q_2) \otimes \chi(q_{\sigma(3)}, Q_3)$ for $\sigma \in S_3$, e.g. $\phi_{g_1} = \chi(q_2, Q_1) \otimes \chi(q_3, Q_2) \otimes \chi(q_1, Q_3)$.

The reason to form the combinations eq. (25)-(25) is that they are idempotent and orthogonal eigenstates of the "charge operators" g_1 and g_2, because of the following relations

$$g_1\phi_A = \phi_A \tag{26}$$

$$g_1\phi_B = \alpha^2\phi_B \tag{27}$$

$$g_1\phi_C = \alpha\phi_C \tag{28}$$

$$g_2\phi_A = \phi_A \tag{29}$$

$$g_2\phi_B = \alpha\phi_B \tag{30}$$

$$g_2\phi_C = \alpha^2\phi_C \tag{31}$$

which follow easily from the properties $g_1^2 = g_2$, $g_1 g_2 = id$ and $g_2^2 = g_1$.

With these building blocks in mind one may now write down the complete formula for each fermion as a specific linear combination of the 24 product states eq (8):

$$\phi^J = \sum_{a,b,c,d=1}^{24} \lambda^J_{abcd} |abcd> \tag{32}$$

where $J = \nu_e, e, ...$ numbers the 24 elementary fermions. The τ-neutrino, for example, corresponds to the overall singlet

$$\begin{aligned} \phi_{\nu_\tau} = \frac{1}{\sqrt{24}} \{ & \phi_{\overline{1234}} + \phi_{\overline{2143}} + \phi_{\overline{3412}} + \phi_{\overline{4321}} + \phi_{\overline{3214}} + \phi_{\overline{2341}} + \phi_{\overline{1432}} + \phi_{\overline{4123}} \\ & + \phi_{\overline{2134}} + \phi_{\overline{1243}} + \phi_{\overline{3421}} + \phi_{\overline{4312}} + \phi_{\overline{3124}} + \phi_{\overline{2314}} + \phi_{\overline{2431}} + \phi_{\overline{1423}} \\ & + \phi_{\overline{4231}} + \phi_{\overline{2413}} + \phi_{\overline{3142}} + \phi_{\overline{1324}} + \phi_{\overline{3241}} + \phi_{\overline{2314}} + \phi_{\overline{4132}} + \phi_{\overline{1423}} \} \end{aligned} \tag{33}$$

and the wave function for the τ-lepton is given by

$$\begin{aligned} \phi_{\tau} = \frac{1}{\sqrt{24}} \{ & \phi_{\overline{1234}} + \phi_{\overline{2143}} + \phi_{\overline{3412}} + \phi_{\overline{4321}} - \phi_{\overline{3214}} - \phi_{\overline{2341}} - \phi_{\overline{1432}} - \phi_{\overline{4123}} \\ & - \phi_{\overline{2134}} - \phi_{\overline{1243}} - \phi_{\overline{3421}} - \phi_{\overline{4312}} + \phi_{\overline{3124}} + \phi_{\overline{2314}} + \phi_{\overline{2431}} + \phi_{\overline{1423}} \\ & - \phi_{\overline{4231}} - \phi_{\overline{2413}} - \phi_{\overline{3142}} - \phi_{\overline{1324}} + \phi_{\overline{3241}} + \phi_{\overline{2314}} + \phi_{\overline{4132}} + \phi_{\overline{1423}} \} \end{aligned} \tag{34}$$

4 A phenomenological Approach to Masses and Charges

As stated, to first approximation all 24 states are identical (symmetric limit). In this limit all masses are equal. Since the symmetry breaking mechanism must have to do with the (new) interaction that keeps the 4 tetrons together, the interaction must have a S_4-breaking part H_X and (using for simplicity nonrelativistic framework) in first order perturbation theory the masses would be calculable as

$$m_{\overline{abcd}} = < abcd | H_X | abcd > \tag{35}$$

One could then use the charge eigenstates eq. (32) to calculate the matrix elements eq. (35). If one further assumes that

$$H_X = V(x_1, x_2, x_3, x_4) = V_{12} + V_{13} + V_{14} + V_{23} + V_{24} + V_{34} \tag{36}$$

with $V_{ij} = q_i q_j V_0(|x_i - x_j|)$, one can easily calculate

$$m_J =< \phi^J|V|\phi^J >= \sum_{abcd,a'b'c'd'} \lambda^*_{abcd}\lambda_{a'b'c'd} < abcd|V|a'b'c'd' > \tag{37}$$

with

$$\begin{aligned} < abcd|V|a'b'c'd' > \;=\;& \delta_{cc'}\delta_{dd'}q_1q_2[\delta_{aa'}\delta_{bb'}J^C_{ab} + \delta_{ab'}\delta_{ba'}J^A_{ab}] \\ & +\delta_{bb'}\delta_{dd'}q_1q_3[\delta_{aa'}\delta_{cc'}J^C_{ac} + \delta_{ac'}\delta_{ca'}J^A_{ac}] \\ & +\delta_{bb'}\delta_{cc'}q_1q_4[\delta_{aa'}\delta_{dd'}J^C_{ad} + \delta_{ad'}\delta_{da'}J^A_{ad}] \\ & +\delta_{aa'}\delta_{dd'}q_2q_3[\delta_{bb'}\delta_{cc'}J^C_{bc} + \delta_{bc'}\delta_{cb'}J^A_{bc}] \\ & +\delta_{aa'}\delta_{cc'}q_2q_4[\delta_{bb'}\delta_{dd'}J^C_{bd} + \delta_{bd'}\delta_{db'}J^A_{bd}] \\ & +\delta_{aa'}\delta_{bb'}q_3q_4[\delta_{cc'}\delta_{dd'}J^C_{cd} + \delta_{cd'}\delta_{dc'}J^A_{cd}] \end{aligned} \tag{38}$$

J^A und J^C are generalizations of exchange and Coulomb integrals

$$J^C_{ab} = \int dx_i^4 d^4x_j V_0(|x_i - x_j|)\phi^*_a(x_i)\phi^*_b(x_j)\phi^*_a(x_i)\phi^*_b(x_j) \tag{39}$$

$$J^C_{ab} = \int dx_i^4 d^4x_j V_0(|x_i - x_j|)\phi^*_a(x_i)\phi^*_b(x_j)\phi^*_a(x_i)\phi^*_b(x_j) \tag{40}$$

They are symmetric in the interchange of a and b, so that besides the 4 unknowns q_i one has 20 unknown integrals.

We do not undertake to examine this further, because apart from the many unknown integrals there is the additional uncertainty about the validity of the nonrelativistic approach. The nonrelativistic picture with a Hamiltonian of the form $H = \sum_i E^i_{kin} + H_X$ with $H_X = V = \sum_{ij} V(|x_i - x_j|)$ is qualitatively nice to give a good overview what states exist. However, it is very unlikely that it works quantitatively correct.

Instead I shall analyze the mass matrix of the fermions as it comes out from their description of the form of linear combinations of tetrahedron states eq. (34). A complete analysis of these states would involve a 24 times 24 mass matrix. We shall not attempt this here but to get an intuition about what can be achieved we split the problem into 3 pieces

- Mass splittings among the weak isospin dublets: in the present model weak isospin states like ν_e and e are described by S_2 wave functions $\phi_\pm$ eq. (17) which are mass eigenstates with masses

$$m_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \end{pmatrix}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{41}$$

  It is then easy to accomodate a mass structure $m_e >> m_{\nu_e}$ namely with a "democratic" mass matrix $m_{11} = m_{12} = m_{21} = m_{22}$.

- Mass splittings among the families: in the present model family states like e, μ and τ are described by Z_3 wave functions $\phi_{A,B,C}$ eq. (25), which are mass eigenstates with masses

$$m_{A,B,C} = \frac{1}{3}\begin{pmatrix} 1 & 1, \alpha^*, \alpha^{*2} & 1, \alpha^{*2}, \alpha^* \end{pmatrix}\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 1, \alpha, \alpha^2 \\ 1, \alpha^2, \alpha \end{pmatrix} \tag{42}$$

  Using the algebraic identity $1 + \alpha + \alpha^2 = 0$ which implies $Im(\alpha) = -Im(\alpha^2)$ and assuming the mass matrix to be symmetrical, it is easy to show that

$$m_A = \frac{1}{3}(tr(m) + 2m_n) \tag{43}$$

$$m_B = m_C = \frac{1}{3}(tr(m) - m_n) \tag{44}$$

where $m_n = m_{12}+m_{13}+m_{23}$ and one immediately sees that by choosing $tr(m) = m_n$ one can obtain a heavy family with $m_\tau = m_A$ and two light ones $m_B = m_C = 0$.

- Mass splittings among the quarks and leptons: in the present model quark and lepton states E and D are distinguished by their behavior under Z_4 and therefore described by the wave functions ϕ_E and $\phi_{D1,2,3}$, eq. (22), which are mass eigenstates with masses

$$m_{E,D_1,D_2,D_3} = \frac{1}{4}\begin{pmatrix} 1,1,1,1 & 1,-i,-1,i & 1,-1,1,-1 & 1,i,-1,-i \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} 1,1,1,1 \\ 1,i,-1,-i \\ 1,-1,1,-1 \\ 1,-i,-1,i \end{pmatrix} \quad (45)$$

The requirement $m_{D_1} = m_{D_2} = m_{D_3}$ of quark masses being color independent directly leads to

$$m_E = \frac{1}{4}(tr(m) + 6m_n) \quad (46)$$

$$m_{D_1} = m_{D_2} = m_{D_3} = \frac{1}{4}(tr(m) - 2m_n) \quad (47)$$

where $m_n = m_{12} + m_{34} = m_{13} + m_{24} = m_{13} + m_{23}$ and one is this way able to accomodate any quark lepton mass ratio one likes.

Notes added:

1. Although this approach knows nothing about the true nature of the tetron interactions and relies solely on the symmetry properties of the tetrahedrons, it can in principle be used to calculate all mass ratios of the fermion spectrum. The basic mass scale is of course set by the strength of the new interaction, but mass ratios can be inferred from symmetry principles.

2. Almost identical results can be obtained on the basis of eq. (35), if one

supposes that the matrix elements transform according to a representation of S_4 or S_3 as described in the appendix. This will not be discussed here.

The above procedure for masses may be applied as well to standard model fermion charges, provided one assumes that each charge C corresponds to an operator C_{op} acting on the linear combinations of states ϕ^J eq. (32) as

$$C_{op}\phi^J = C_J\phi^J \tag{48}$$

In the left-right symmetric standard model $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ there are the following charges: Y_{B-L} (B-L charge), C_8 and C_3 (color charges) and T_{3L} and T_{3R} for weak isospin.

- Starting with weak isospin dublets and using that weak isospin states are described in the present model by S_2 wave functions $\phi_\pm$ eq. (17) with eigenvalues $\pm\frac{1}{2}$ we require

$$\pm\frac{1}{2} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{49}$$

  We therefore find that

$$T_{3op} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{50}$$

  and the analysis is identical for left and righthanded fermions.

- Going next to B-L and to the color charges we have for the eigenvalues of a quark-lepton quadruplet L and Q the following condition:

$$C(L, Q_1, Q_2, Q_3) = \frac{1}{4}\begin{pmatrix} 1,1,1,1 & 1,-i,-1,i & 1,-1,1,-1 & 1,i,-1,-i \end{pmatrix}$$
$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}\begin{pmatrix} 1,1,1,1 \\ 1,i,-1,-i \\ 1,-1,1,-1 \\ 1,-i,-1,i \end{pmatrix} \tag{51}$$

where on the left hand side there are the standard model values $C_8(L) = 0$, $C_8(Q_1) = 1$, $C_8(Q_2) = -2$, $C_8(Q_3) = 1$ and similarly $C_3(L) = 0$, $C_3(Q_1) = 1$, $C_3(Q_2) = 0$, $C_3(Q_3) = -1$ and $Y_{B-L}(L) = -\frac{1}{2}$, $Y_{B-L}(Q_1) = Y_{B-L}(Q_2) = Y_{B-L}(Q_3) = \frac{1}{6}$. With a little algebra one obtains

$$Y_{B-L} = -\frac{1}{3}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{52}$$

$$C_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \tag{53}$$

$$C_8 = \frac{1}{2}\begin{pmatrix} 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \\ -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \tag{54}$$

These charge operators can be extended to 24 times 24 matrices acting on the fermion wave functions eq. (34). They cannot be traced to charges of a single tetron, but correspond to properties of the tetrahedron wave functions ϕ_σ. As will be shown, there is a direct relation to the construction of vector boson states in the next section.

5 The Model for Vector Bosons

Vector Bosons are bound states of 8 tetrons which arise when 2 of the fermion-tetrahedrons $\phi_\sigma = \phi_{\overline{abcd}}$ and $\phi_{\sigma'} = \phi_{\overline{a'b'c'd'}}$ of eq. (8) approach each other and a and a', b and b', c and c' and d and d' interact in such a way that a cube is formed. More precisely, I am talking about a tetrahedron and an anti-tetrahedron, where a and a', b and b', c and c' and d and d' sit on opposite corners of the cube. Note that we are dealing with Spin 1 objects here because of the spin $\frac{1}{2}$ nature of the tetrahedrons discussed in section 2. When forming fermion-antifermion products one should therefore in principle write $\bar{\phi}_{\sigma'}\gamma_\mu\phi_\sigma$. To keep things manageable, however, the Dirac structure will not be made explicit in the formulas below. Instead I shall simply write $\phi^*_{\sigma'}\phi_\sigma$.

In the symmetric limit, in which all tetrons are identical, there are 24×24 identical cube states. When the symmetry is broken with charges q_a ... $q_{d'}$, Q_a ... $Q_{d'}$, one has 24^2 different states generated by applying the elements of $S_4' \times S_4$ on $\phi^*_{\sigma'}\phi_\sigma$. One is thus confronted with a lot of different tetrahedron-antitetrahedron bound states which mediate a lot of interactions like inter family interactions, leptoquarks and so on, which one does not want.

In order to reduce the number of vector boson states there must be a mechanism which in the process of vector boson formation from fermions (i.e. the tetrahedron and anti-tetrahedron approaching each other) restores parts of the symmetry like, for example, the Z_3-(family)-part, so that there are no interaction particles associated with Z_3. Instead Z_3 must become a symmetry of the cube which transforms e.g. the W^+ arising from $\tau^+ + \nu_\tau$ to a W^+ decaying into $e^+ + \nu_e$. A similar rule holds for the transformations between quarks and leptons forbidding the existence of leptoquarks, i.e. lepton number violating processes, and at the same time allowing W^+-mediated transitions e.g. between $\bar{b} + t$ and $\tau^+ + \nu_\tau$.

The background behind this is a nonexistence

THEOREM: The following statements are equivalent:

- the weak interactions are universal
- leptoquark and interfamily interactions do not exist

This theorem seems quite obvious on an abstract level and in particular in the present model because one needs interfamily and leptoquark transitions as symmetries and not as interactions. It should be noted however that in principle the symmetries of vector bosons under tetron transformations could be larger than given by the theorem.

Later I am going to explicitly identify which transitions are symmetries for vector bosons and which ones are not. This will be done by considering step by step the possible fermion-antifermion states and comparing them to the observed vector bosons.

Before doing this, let me discuss in some detail, what might be the cause of the increased symmetry, i.e. the universality of the weak interaction, on the tetron level. One may speculate that it is caused by a partwise annihilation of tetron charge clouds q_i so that some corners of the cube become indistinguishable and the cube symmetrical under Z_3 and leptoquark transformations. As soon as this happens, these symmetries can be interpreted as ordinary rotation symmetries in space (e.g. for Z_3: $\pm 2\pi/3$ rotations about the 3 body diagonals of the cube). This annihilation may affect, for example, the 'front' triangle of the tetrahedron which is approaching a corresponding 'front' of an anti-tetrahedron. Stripped off of 3 of their clouds, e.g. q_a, q_b and q_d the tetrahedron will then look like $\chi(Q_1)\otimes\chi(Q_2)\otimes\chi(q_c, Q_3)\otimes\chi(Q_4)$ and similarly for the anti-tetrahedron and the resulting cube will be symmetric under an S_3 and in particular under interchanges $(Q_1 \leftrightarrow Q_2)$ and $(Q_1 \leftrightarrow Q_4)$ which according to table 1 transform the families into each other. Note however, that this picture is at best heuristic as was the assignment of particle states in table 1. The point is that according to sect. 3 fermion states are linear combinations of permutation states and this complicates the situation. Therefore, instead of giving handwaving arguments I will now consider explicitly the possible fermion-antifermion states.

I start with the electroweak sector and want to represent $W^\pm$ and Z by suitable combinations $\phi^*_{\overline{a'b'c'd'}}\phi_{\overline{abcd}}$. A simple formula can be obtained from the wave functions eq. (17):

$$\begin{aligned} W^- &= {}'\bar\nu_l\gamma_\mu l' = \phi_+^*\phi_- \\ &= [\chi^*(q_1',Q_1')\otimes\chi^*(q_3',Q_3')+\chi^*(q_1',Q_3')\otimes\chi^*(q_3',Q_1')] \\ &\quad \otimes[\chi(q_1,Q_1)\otimes\chi(q_3,Q_3)-\chi(q_1,Q_3)\otimes\chi(q_3,Q_1)] \end{aligned} \tag{55}$$

and

$$Z = {}'\bar\nu_l\gamma_\mu\nu_l - \bar l\gamma_\mu l' = |\phi_+|^2-|\phi_-|^2 \tag{56}$$

where by Z I actually do not mean the physical Z-boson but what is usually called W_3. I will from now on leave out the primes in the arguments implicitly understanding that whereever a wave functions with an asterics appears it should get primed arguments. Then, under very moderate assumptions on the product $\otimes$, the above formulae may be simplified to

$$\begin{aligned} W^\pm &= |\chi(q_1,Q_1)\chi(q_3,Q_3)|^2-|\chi(q_1,Q_3)\chi(q_3,Q_1)|^2 \\ &\pm [\chi^*(q_1,Q_1)\chi^*(q_3,Q_3)\chi(q_1,Q_3)\chi(q_3,Q_1)-c.c.] \\ Z &= 2[\chi^*(q_1,Q_1)\chi^*(q_3,Q_3)\chi(q_1,Q_3)\chi(q_3,Q_1)+c.c.] \end{aligned} \tag{57}$$

These results can easily be generalized to make the universality of $W^\pm$ and Z on the family level explicit. Namely, using the representation (57) of the wave functions the Z_3-symmetry of the weak interactions implies

$$\begin{aligned} W^- &= \{|\phi_{id}|^2-|\phi_{u_0}|^2+\phi_{id}^*\phi_{u_0}-\phi_{u_0}^*\phi_{id}\} \\ &= \{|\phi_{g_1}|^2-|\phi_{u_1}|^2+\phi_{g_1}^*\phi_{u_1}-\phi_{u_1}^*\phi_{g_1}\} \\ &= \{|\phi_{g_2}|^2-|\phi_{u_2}|^2+\phi_{g_2}^*\phi_{u_2}-\phi_{u_2}^*\phi_{g_2}\} \\ Z &= 2[\phi_{id}^*\phi_{u_0}+\phi_{u_0}^*\phi_{id}] \\ &= 2[\phi_{g_1}^*\phi_{u_1}+\phi_{u_1}^*\phi_{g_1}] \\ &= 2[\phi_{g_2}^*\phi_{u_2}+\phi_{u_2}^*\phi_{g_2}] \end{aligned} \tag{58}$$

Note that instead of eq. (17) or (16) or (25) one should of course use full combinations like eqs. (33) and (34) as factors to be used in the representation of vector bosons. In this case the restrictions from the nonexistence theorem become more complicated in shape, but the principle does not change.

The validity of the nonexistence theorem, i.e. absence of interfamily interactions, can also be explicitly verified by showing that terms like $\bar\nu_e\mu$, $\bar\nu_\mu\tau$ and $\bar\nu_\tau e$ which represent the Z_3-symmetric form of interfamily interactions identically vanish.

In a similar fashion as eq. (56) one can use the representation of leptons and quarks eqs. (13) and (16) to write down a formula for the photon:

$$\begin{aligned}\gamma &= {}'3\bar\nu_\tau\gamma_\mu\nu_\tau - \bar t_1\gamma_\mu t_1 - \bar t_2\gamma_\mu t_2 - \bar t_3\gamma_\mu t_3' \\ &= 3|\phi_{\overline{1234}} + \phi_{\overline{2341}} + \phi_{\overline{3412}} + \phi_{\overline{4123}}|^2 - |\phi_{\overline{1234}} + i\phi_{\overline{2341}} - \phi_{\overline{3412}} - i\phi_{\overline{4123}}|^2 \\ &\quad -|\phi_{\overline{1234}} - \phi_{\overline{2341}} + \phi_{\overline{3412}} - \phi_{\overline{4123}}|^2 - |\phi_{\overline{1234}} - i\phi_{\overline{2341}} - \phi_{\overline{3412}} + i\phi_{\overline{4123}}|^2\end{aligned} \tag{59}$$

where by 'photon' I always mean the $U(1)_{B-L}$ gauge boson, as well as for gluons like

$$\begin{aligned}g_3 &= {}'\bar t_1\gamma_\mu t_1 - \bar t_2\gamma_\mu t_2' \\ &= |\phi_{\overline{1234}} + i\phi_{\overline{2143}} - \phi_{\overline{3412}} - i\phi_{\overline{4321}}|^2 - |\phi_{\overline{1234}} - i\phi_{\overline{2143}} - \phi_{\overline{3412}} + i\phi_{\overline{4321}}|^2\end{aligned} \tag{60}$$

or

$$\begin{aligned}g_8 &= {}'3\bar\nu_\tau\gamma_\mu\nu_\tau - \bar t_1\gamma_\mu t_1 - \bar t_2\gamma_\mu t_2 - \bar t_3\gamma_\mu t_3' \\ &= 2|\phi_{\overline{1234}} - \phi_{\overline{2143}} + \phi_{\overline{3412}} - \phi_{\overline{4321}}|^2 - |\phi_{\overline{1234}} + i\phi_{\overline{2143}} - \phi_{\overline{3412}} - i\phi_{\overline{4321}}|^2 \\ &\quad -|\phi_{\overline{1234}} - i\phi_{\overline{2143}} - \phi_{\overline{3412}} + i\phi_{\overline{4321}}|^2\end{aligned} \tag{61}$$

and analogously for leptoquarks

$$\begin{aligned}{}'\bar l\gamma_\mu q_1' &= [\phi_{\overline{1234}} + \phi_{\overline{2341}} + \phi_{\overline{3412}} + \phi_{\overline{4123}}]^*[\phi_{\overline{1234}} + i\phi_{\overline{2143}} - \phi_{\overline{3412}} - i\phi_{\overline{4321}}] \\ {}'\bar l\gamma_\mu q_2' &= [\phi_{\overline{1234}} + \phi_{\overline{2341}} + \phi_{\overline{3412}} + \phi_{\overline{4123}}]^*[\phi_{\overline{1234}} - \phi_{\overline{2143}} + \phi_{\overline{3412}} - \phi_{\overline{4321}}] \\ {}'\bar l\gamma_\mu q_3' &= [\phi_{\overline{1234}} + \phi_{\overline{2341}} + \phi_{\overline{3412}} + \phi_{\overline{4123}}]^*[\phi_{\overline{1234}} - i\phi_{\overline{2143}} - \phi_{\overline{3412}} + i\phi_{\overline{4321}}]\end{aligned} \tag{62}$$

In order that the leptoquarks vanish in accord with the nonexistens theorem and with phenomenology we impose the symmetry of the vector boson states under the transformation $\overline{3412} = (1 \leftrightarrow 3, 2 \leftrightarrow 4)$. A straightforward calculation yields $\bar{l}\gamma_\mu q_1 = \bar{l}\gamma_\mu q_2 = \bar{l}\gamma_\mu q_3 = 0$ which is the desired result. Note that the absence of leptoquarks indicates that the effective gauge group is not $SU(4)$ but $U(1)_{B-L} \times SU(3)_c$. There is no room in this model for exotic grand unified vector bosons.

Imposing the symmetry $\overline{3412}$ on vector bosons also allows to simplify the formulae for photon and gluons

$$\gamma = 8\phi^*_{\overline{1234}}[\phi_{\overline{2341}} + \phi_{\overline{3412}} + \phi_{\overline{4123}}] + 8\phi^*_{\overline{2341}}[\phi_{\overline{1234}} + \phi_{\overline{3412}} + \phi_{\overline{4123}}] \quad (63)$$

$$g_3 = 4i\phi^*_{\overline{1234}}[\phi_{\overline{2341}} - \phi_{\overline{4123}}] + 4i\phi^*_{\overline{2341}}[\phi_{\overline{3412}} - \phi_{\overline{1234}}] \quad (64)$$

$$g_8 = 4\phi^*_{\overline{1234}}[2\phi_{\overline{3412}} - \phi_{\overline{2341}} - \phi_{\overline{4123}}] + 4\phi^*_{\overline{2341}}[2\phi_{\overline{4123}} - \phi_{\overline{1234}} - \phi_{\overline{3412}}] \quad (65)$$

This result should be compared to the charge matrices eqs. (53)-(54). It is interesting to note that the diagonal terms $|\phi_{\overline{1234}}|^2$ and $|\phi_{\overline{2341}}|^2$ do not arise in connection with the 'diagonal' gluons g_3 and g_8. This fact is in accord with the zeros on the diagonals of the matrices eqs. (54) and(54). They only contribute to the nondiagonal gluons, as one can easily convince oneself. There is a caveat about the photon: it will in fact turn out to have a somewhat more complicated representation than given by eq. (63), see later.

In summary we have analyzed the posible interactions between tetrahedron and antitetrahedron states and shown which transformations should be symmetries of the resulting vector boson state and which ones should not. Namely, symmetries are $\overline{3412} = (1 \leftrightarrow 3, 2 \leftrightarrow 4)$ and the Z_3-family transformations, whereas $\overline{3214} = (1 \leftrightarrow 3)$ (for the weak interactions), and

$\overline{4321} = (1 \leftrightarrow 4, 2 \leftrightarrow 3)$ and $\overline{2143} = (1 \leftrightarrow 2, 3 \leftrightarrow 4)$ (for the strong interactions) are broken. As a consequence, the original 24^2 $S_4' \times S_4$ tetrahedron-antitetrahedron states get reduced to

- 8 gluon states generated by

$$S_4'/(Z_2' \diamond S_3') \times S_4/S_3 = Z_2' \times Z_4 \tag{66}$$

  where the first factor corresponds to the symmetry $\overline{3412}$ and the second factor is due to the fact that the gluons act homogeneously on the 6 orbits corresponding to family symmetry and weak isospin. In more concrete terms the 8 real degrees of freedom are given by

$$\begin{aligned} &|\phi_{\overline{1234}}|^2 \\ &|\phi_{\overline{2341}}|^2 \\ \phi^*_{\overline{1234}}\phi_{\overline{3412}} &= [\phi^*_{\overline{1234}}\phi_{\overline{3412}}]^* \\ \phi^*_{\overline{2341}}\phi_{\overline{4123}} &= [\phi^*_{\overline{2341}}\phi_{\overline{4123}}]^* \\ \phi^*_{\overline{1234}}\phi_{\overline{2341}} &= \phi^*_{\overline{2341}}\phi_{\overline{3412}} \\ \phi^*_{\overline{1234}}\phi_{\overline{4123}} &= [\phi^*_{\overline{2341}}\phi_{\overline{3412}}]^* \end{aligned} \tag{67}$$

  precisely the objects which appear in eqs. (63), (64) and (65) and for the 'nondiagonal' gluons.

- 4 states γ, $W^\pm$ and Z generated by

$$S_4'/A_4' \times S_4/(Z_4 \diamond Z_2) = Z_2' \times Z_2 \tag{68}$$

  where $A_4 \subset S_4$ is the group of even permutations in S_4. The first factor in (68) corresponds to the symmetry breaking under odd permutations like $(1 \leftrightarrow 3)$ whereas the second factor is due to the 12 orbits in the family and color group. In more concrete terms the resulting 4 real degrees of freedom are given by

$$
\begin{aligned}
&|\phi_{\overline{1234}}|^2 \\
&|\phi_{\overline{3214}}|^2 \\
\phi^*_{\overline{1234}}\phi_{\overline{3214}} &= [\phi^*_{\overline{3214}}\phi_{\overline{1234}}]^* \qquad (69)
\end{aligned}
$$

objects, which have already appeared, in a somewhat different notation, in eqs. (55) and (58). Note there is one degree of freedom which was not used in those equations, namely the 'singlet'

$$|\phi_{id}|^2 + |\phi_{u_0}|^2 = |\phi_{\overline{1234}}|^2 + |\phi_{\overline{3214}}|^2 \qquad (70)$$

and this corresponds to the photon. In fact for the photon, eq. (61) is not the whole story, because the factorization of the photon interaction is not as simple as for the strong and weak case. Eq. (61) has to be extended using eqs. (19)-(22) to include both isospin sectors of quarks and leptons.

One item has been spared out so far: the question of spin. In fact, most of the preceeding analysis can be carried out for lefthanded and righthanded fermions separately. This way one naturally ends up with separate weak bosons for left and right. For the weak interactions this is okay, because we are expecting an effective $SU(2)_L \times SU(2)_R$ gauge theory. Photon and gluons, however, behave differently. They are identical for left- and righthanded fermions, i.e. they interact vectorlike. The question then immediately arises: what makes them peculiar?

The full answer to this question will be given in a forthcoming paper [9]. Actually, the vector boson formation according to eqs. (59)ff has to be modified when spin is taken into account. It turns out that helicity flips and Z_4-transformations are independent operations for fermions but not for vector bosons. Helicity transformations essentially rob the axialvector degrees of freedom, so that only vector couplings remain. The argument is in principle similar as in the case of the above nonexistence theorem which relied on the universality of weak interactions.

As shown in [9] helicity flips are related to the transformation $\overline{3412}$ which was imposed as a symmetry on gluons, leptoquarks and photon in eqs. (63)-(65), and this transformation will be shown to leave the vector combination $\bar{\phi}_{\sigma',L}\gamma_\mu\phi_{\sigma,L} + \bar{\phi}_{\sigma',R}\gamma_\mu\phi_{\sigma,R}$ invariant but not the axial vector combination $\bar{\phi}_{\sigma',L}\gamma_\mu\phi_{\sigma,L} - \bar{\phi}_{\sigma',R}\gamma_\mu\phi_{\sigma,R}$.

6 Conclusions

In conclusion we have here a scheme which accomodates all observed fermions and vector bosons. In addition it relates the number of these states in an obvious and natural way to the number of space dimensions, because the tetrahedron with 4 fundamental constituents is the minimal complex to build up 3 dimensions. With 3 constituents, for example, one would live in a 2-dimensional world with bound states of triangles which would yield only 6 bound permutation states instead of 24.

As we have seen, there are several other respects like parity violation and mass hierarchies in which the present model goes a step further in understanding than standard gauge theories. The real challenge will of course be to understand the nature of the tetron interactions and to write them in a renormalizable form.

About possible experimental tests: in the low energy limit the tetrahedron goes over into a point, i.e. it becomes an ordinary pointlike fermion. Increasing the energy one should be able to dissolve its spatial extension which will show up e.g. in the form of non-Dirac-like form factors. The question then is: how small are the extensions of the tetrahedron? This question is difficult to answer and is related to the problem of the strength and nature of the superstrong interaction which binds the tetrons together. Certainly there will be a correspondence between the tetrahedron extension and the new coupling constant which is a new fundamental constant of nature from which most of the couplings and masses known in particle physics will be derived.

7 Appendix: Group and Representation Structure of $T_d = S_4$

S_4 is the group of permutations of 4 objects and isomorphic to the symmetry group T_d of an equilateral tetrahedron. Among the subgroups of S_4 there are A_4, Z_3, S_2, S_3, Z_4 and the so-called *Kleinsche Vierergruppe* K consisting of permutations $\overline{1234}$, $\overline{2143}$, $\overline{3412}$ and $\overline{4321}$ which is isomorphic to $Z_2 \times Z_2$. It is is the smallest noncyclic group and describes the symmetries of a rectangle in 2 dimensions. Z_3 is the cyclic group of 3 elements or equivalently the subgroup of S_3 corresponding to the even permutations or equivalently the group generated by $\pm 2\pi/3$ rotations of the plane.

S_4 can be written as $Z_4 \diamond S_3$ where $S_3 = Z_3 \diamond S_2$ is the point symmetry group of an equilateral triangle. Correspondingly there are 4 orbits of S_3 and 6 orbits of Z_4 in S_4. In the present model the 4 degrees of Z_4 correspond to a lepton and a quark with three colors. Z_3 is the family group and S_2 roughly corresponds to weak isospin.

A short review of the representations of S_4 is in order: for a finite group the number of irreducible representations is given by the number of conjugacy classes (=5 in this case). Apart from the trivial representation A_1 there is the totally antisymmetric representation A_2, which assigns 1 to all even and -1 to every odd permutation. A_1 and A_2 are of dimension 1. There are two 3-dimensional representations usually called T_1 and T_2, T_1 describing the action of the permutation group on the tetrahedron. T_2 differs from T_1 by a minus sign for odd permutations. Finally there is a 2-dimensional representation E which may be considered as the trivial extension of a corresponding 2-dimensional representation of S_3.

One may use the representations E and T_1 in the context of the symmetry breaking, because in first order approximation the representation matrices may be considered as state vectors (in the case of E an average 'family state vector') of the fermions. This has to do with the fact that the permutations (and its representations) may be considered as ladder operators which generate all states $\overline{abcd}$ from the ground state $\overline{1234}$. These unbroken states can then be used, in the spirit of first order perturbation theory, to derive mass and charge relations for the fermions. (I have not used this approach in the present paper, because I found a more elegant way to derive these relations.)

The covering group $\tilde{S}_4$ of S_4 is embedded in $SU(2)$ just as S_4 is in $SO(3)$. It is important for assigning Spin to a tetrahedron. Apart from those representations which are extensions from S_4 to $\tilde{S}_4$ it has the following representations: a 2-dimensional representation G_1 which is the restriction of the fundamental representation of SU_2 to the symmetry transformations of a tetrahedron in much the same way as T_1 is the restriction of the fundamental representation of SO_3. Then there is a representations G_2 obtained from G_1 like T_2 is obtained from T_1 and a 4-dimensional (spin 3/2) representation H.

References

[1] E.S. Abers and B.W. Lee, Phys.Rep. 9C (1973) 1

[2] for a review on neutrino masses see e.g. G. Altarelli, Proc. Rencentres de La Thuile, 2005

[3] B. Lampe, hep-ph/9512276 (1995)

[4] R.N. Mohapatra and J.C. Pati, Phys. Rev. D11 (1975) 566, 2558

[5] J.C. Pati and A. Salam, Phys. Rev. D10 (1974) 275

[6] H.Fritzsch and P. Minkowski, Ann. Phys. 93 (1975) 193

[7] B.L. van der Waerden, Algebra, Springer Verlag

[8] B. Lampe, hep-ph/9810417 (1998)

[9] B. Lampe, in preparation