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Some Remarks on the Tetron Spin Problem

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Abstract

The possible spatial transformation properties of tetrons are discussed.

1 Introduction

In recent papers [1, 2] the inner symmetries of the tetron model have been discussed on the basis of the known representations A_1, A_2, E, T_1 and T_2 of the permutation group S_4.

Correspondingly, a S_4-'flavor' permutation index taking the values a,b,c and d has been introduced and suggested to be part of a larger continuous flavor symmetry SU(4). More concretely, it has been shown that the 24 S_4 representation states can be considered as part of the 256 SU(4)-states of the tensor product

$$\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} = 3\times\mathbf{45}(\mathbf{T_1}) + 3\times\mathbf{15}(\mathbf{T_2}) + 2\times\mathbf{20}(\mathbf{E}) + \mathbf{35}(\mathbf{A_1}) + \mathbf{1}(\mathbf{A_2}) \quad (1)$$

where one finds in brackets, which kind of S_4 symmetry functions are contained in the corresponding SU(4) representations, giving rise to the observed 3-generation spectrum of quarks (T_1 and T_2) and leptons (A_1, A_2 and E).

One then needs an exclusion principle which demands, that among the 256 SU(4)-states only the 24 S_4 representation states are physical. We shall see later how this exclusion principle may be related to the spatial transformation properties of tetrons which will be discussed now.

2 The Tetron Spin Problem

Eq. (1) presupposes a constituent picture where quarks and leptons are built from 4 tetron 'flavors' a, b, c and d. In this secenario, one immediately faces the problem, how the spin-$\frac{1}{2}$ fermion behavior of quarks and leptons arises on a microscopic level.

In this note I will present the framework, in which this problem should be solved. I will consider spatial transformations only. The extension to Minkowski space will be worked out in a separate publication [3].

Let me start with a few well-known facts about half-integer spin: in a physical experiment one cannot distinguish between states which differ by a complex phase. Therefore, in addition to ordinary representations one may include projective, half-integer spin representations of the rotation group SO(3), and also of its $T_d = S_4$ subgroup[1]. These are true representations of the corresponding covering groups SU(2) and $\tilde{S}_4$, respectively.

To solve the tetron spin problem I suggest to give up the requirement of continuous rotation symmetry and assume that tetrons live and interact in microscopical environments, in which only permutation symmetry survives. The latter is much less restrictive than rotational SO(3), because the idea of rotation assumes concepts of angle and length, which may be obstacled by quantum fluctuations when approaching the Planck scale. In contrast, the idea of permutation merely presupposes the more fundamental principle of identity. This is why permutation groups may enter theoretical physics at finer levels of resolution and higher energies than the Lorentz group. Tetrons may be more basic than spinors.

I call this assumption the 'spatial permutation hypothesis'. It amounts to introducing a second permutation index taking values 1,2,3 and 4 (in addition to the flavor index a,b,c and d) and being responsible for the spatial ('spin') transformation behavior of tetrons and its compound states.

It is true that the phenomenological observation of 24 quarks and leptons and

[1] T_d is the rotation symmetry group of a regular tetrahedron. It is a subgroup of $O(3)$ and isomorphic to S_4. T_d is also isomorphic to the octahedral group O, i.e. the group of proper rotations of a cube which is a subgroup of SO(3).

their interactions imply a permutation principle only on the level of inner symmetries (as in eq. (1)). However, the assumption of 4 different tetron 'spins' within a fermion bound state comes closest to the original intuition of a spatial tetrahedral structure as discussed in ref. [1] where a generic ansatz for the composite wave function $a_i b_j c_k d_l$ with $i, j, k, l \in \{1, 2, 3, 4\}$ has been proposed.

As a consequence of the spatial permutation hypothesis a new type of particle statistics will arise (called *tetron statistics*) which differs from Fermi and Bose statistics and is related to the exclusion principle in the flavor sector as formulated above.

3 The Details

According to the spatial permutation hypothesis, the spin part of a 4-particle fermionic compound state should transform according to a (projective) representation of S_4. Besides the ordinary representations A_1, A_2, E, T_1 and T_2 there are 3 irreducible projective representations (representations of the covering group $\tilde{S}_4$), namely G_1, G_2 and H of dimensions 2, 2 and 4, respectively [4]. The sum 4+4+16 of the dimensions squared accounts for the 24 additional elements due to the Z_2 covering of S_4. Among them, G_1 uniquely corresponds to spin-$\frac{1}{2}$, i.e. is obtained as the restriction of the fundamental SU(2) representation to $\tilde{S}_4$. Similarly, H can be obtained from the spin-$\frac{3}{2}$ representation of SU(2), whereas G_2 is obtained from G_1 by reversing the sign for odd permutations and contains contributions from spins larger than $\frac{3}{2}$.

For the understanding of the following arguments a short digression on quaternions and its usefulness for describing nonrelativistic spin-$\frac{1}{2}$ fermions will be helpful:

Quaternions are a non-commutative extension of the complex numbers and play a special role in mathematics, because they form one of only three finite-dimensional division algebra containing the real numbers as a subalgebra. (The other two are the complex numbers and the octonions.) As a vector space they are generated by 4 basis elementes 1, I, J and K which fulfill $I^2 = J^2 = K^2 = IJK = -1$, where K can be obtained as a product $K = IJ$ from I and J. Quaternions are non-commutative in the sense IJ=-JI. Any quaternion q has an expansion of the form

$$\begin{aligned} q &= c_1 + Jc_2 \\ &= r_1 + Ir_2 + Jr_3 + Kr_4 \end{aligned} \tag{2}$$

with real r_i and complex $c_1 = r_1 + Ir_2$ and $c_2 = r_3 - Ir_4$.

There is a one-to-one corresponence between unit quaternions q_0 and SU(2) transformation matrices, because the latter are necessarily of the form $(\alpha, \beta; -\beta^*, \alpha^*)$ with complex α and β fulfilling $|\alpha|^2 + |\beta|^2 = 1$, and can be written as $q_0 = \alpha + J\beta$. Therefore, the action of SU(2) matrices on spinor fields (c_1, c_2) (c_1 with spin up and c_2 with spin down) can in quaternion notation be rewritten as:

$$c_1 + Jc_2 \rightarrow (\alpha + J\beta)(c_1 + Jc_2) \tag{3}$$

For example the unit quaternions I and J corresponding to rotations by π about the x and y-axis amount to $c_1 \rightarrow Ic_1, c_2 \rightarrow -Ic_2$ and $c_1 \rightarrow -c_2, c_2 \rightarrow c_1$, respectively. For a general SU(2) transformation one has $c_1 \rightarrow \alpha c_1 - \beta^* c_2$ and $c_2 \rightarrow \alpha^* c_2 + \beta c_1$, from which e.g. the antisymmetric tensor product combination $c_1 c_2' - c_2 c_1'$ can be shown to be rotationally invariant (spin 0).

To describe spin-$\frac{1}{2}$ bound states one should use the symmetry function of the representation G_1. This function will also be called G_1 in the following and can be given as linear combination of the G_1 representation matrices (=unit quaternions):

$$
\begin{aligned}
G_1 &= g(1,2,3,4) + Ug(2,3,1,4) + U^2 g(3,1,2,4) \\
&+ Ig(2,1,4,3) + Sg(3,2,4,1) + R^2 g(1,3,4,2) \\
&+ Jg(3,4,1,2) + Rg(1,4,2,3) + T^2 g(2,4,3,1) \\
&+ Kg(4,3,2,1) + Tg(4,1,3,2) + S^2 g(4,2,1,3) \\
&+ \frac{I+K}{\sqrt{2}} g(3,2,1,4) + \frac{I-J}{\sqrt{2}} g(1,3,2,4) + \frac{J+K}{\sqrt{2}} g(2,1,3,4) \\
&+ \frac{1-J}{\sqrt{2}} g(2,3,4,1) + \frac{1-K}{\sqrt{2}} g(3,1,4,2) + \frac{J-K}{\sqrt{2}} g(1,2,4,3) \\
&+ \frac{I-K}{\sqrt{2}} g(1,4,3,2) + \frac{1+K}{\sqrt{2}} g(2,4,1,3) + \frac{1+I}{\sqrt{2}} g(3,4,2,1) \\
&+ \frac{1+J}{\sqrt{2}} g(4,1,2,3) + \frac{I+J}{\sqrt{2}} g(4,2,3,1) + \frac{1-I}{\sqrt{2}} g(4,3,1,2) \qquad (4)
\end{aligned}
$$

where $R = \frac{1}{2}(1-I-J-K), S = \frac{1}{2}(1-I+J+K), T = \frac{1}{2}(1+I-J+K)$ and $U = \frac{1}{2}(1+I+J-K)$. One can see explicitly from this equation, which S_4 permutation $\overline{ijkl}$ is represented in G_1 by which quaternion, because the corresponding quaternion appears as a coefficient of g(i,j,k,l). For example, the permutation $\overline{2341}$ is represented by $\pm(1-J)/\sqrt{2}$, and so on. In other words, the quaternion coefficients $1, I, J, K, (I+K)/\sqrt{2}, ..., R, S, T, ...$ in this equations represent the elements of $\tilde{S}_4$[2].

[2]While $\tilde{S}_4$ itself can be shown to make up the inner shell of D_4-lattices [5], the first half of coefficients in eq. (4) represent even permutations corresponding to $\tilde{A}_4$ which is sometimes called the 'binary tetrahedral group', and generates the F_4 lattice also called the ring of Hurwitz integers (=quaternions with half integer coefficients). The Hurwitz quaternions form a maximal order (in the sense of ring theory) in the division algebra of quaternions with rational components. This accounts for its importance. For example restricting to integer lattice points, which seems a more obvious candidate for the idea of an integral quaternion, one does not get a maximal order and is therefore less suited for developing a theory of left ideals as in algebraic number theory. What Hurwitz realized, was that his definition of integral quaternions is the better one to operate with.

Due to the 2-fold covering of S_4 each of the real functions $g(i,j,k,l)$ in eq. (4) with its 24 terms is in fact a difference $p(i,j,k,l) - m(i,j,k,l)$ so as to obtain the 48 terms needed for a symmetry function of $\tilde{S}_4$.

Eq. (4) should be considered as the spin factor of the 4-tetron bound state, whereas the A_1, A_2, E, T_1 and T_2-functions of the ordinary S_4 representations account for the flavor factor. In fact, working out the quaternion multiplications in eq. (4) and using $K = IJ$ one obtains a representation of the form $G_1 = c_1 + Jc_2$ with c_1 and c_2 decribing the 2 spin directions of the compound fermions, cf eq. (3). Mathematically, the appearance of 2 complex functions c_1 and c_2 in eq. (4) is merely expression of the fact that for the 2-dimensional representation G_1 4 real(=2 complex) symmetry functions can be constructed, which in eq. (4) are combined in one quaternion function.

Eq. (4) therefore describes a decent fermion state which transforms in the standard way, cf. eq. (3). On the other hand, eq. (4) also inherits the spatial permutation hypothesis (i.e. giving up full SU(2) rotational invariance on the tetron level) in that the function G_1 naturally reacts like a (projective) S_4 representation under permutations of $i,j,k,l \in \{1,2,3,4\}$.

Since it is not possible to build a spin-$\frac{1}{2}$ G_1 state as a 4-tensor product similar to eq. (1), the picture followed here is a sort of molecular approach where one starts with a fixed spatial tetrahedral configuration with 4 distinct permutation 'spin' indices $i,j,k,l \in \{1,2,3,4\}$ which according to the spatial permutation hypothesis must transform according to G_1. Its reaction under permutations (T_d transformations) of i,j,k,l is dictated by the spatial permutation hypothesis, whereas the behavior under full rotational SU(2) is obtained from the requirement that the compound state must be a fermion.

If one wants to go beyond this understanding one should look for possible transformation properties of the spatial permutation indices i,j,k,l in a tensor product, which mimics the behavior of G_1. Since this cannot be obtained within the usual framework of universal Z_2 coverings, one has to consider e.g. octonion Z_4 extensions of the rotation group. In such a framework $g(i,j,k,l)$ (or alternatively p and m) are the functions which should be interpreted as tensor products of tetrons of the generic form

$$g(i,j,k,l) = a_i \otimes b'_j \otimes c''_k \otimes d'''_l \tag{5}$$

where a,b,c,d are the tetron 'flavor' and i,j,k,l their 'spin' indices, so that the complete spin and flavor wave function of quarks and leptons can be written as

$$\begin{aligned} a_1 \otimes b'_2 \otimes c''_3 \otimes d'''_4 \quad &+ \quad b_1 \otimes c'_2 \otimes a''_3 \otimes d'''_4 + ... \\ I a_2 \otimes b'_1 \otimes c''_4 \otimes d'''_3 \quad &+ \quad ... \\ ... \end{aligned} \tag{6}$$

Here in the rows the tetron flavor indices a,b,c,d are permutated in order to obtain the appropriate flavor combination (A_1 of S_4 as an example, for the A_2, T_1 etc flavor representations G_2 and H will come into play), whereas in the columns the tetron spin indices i,j,k,l are permutated in order to obtain the G_1 spin combination.[3]

Very important: eq. (4) reflects the statistical behavior of a 4-tetron conglomerate unter permutations of its components. This behavior has a certain similarity to that of fermions but is certainly not identical. While conglomerates of fermions usually transform with the totally antisymmetric representation (like A_2), tetrons go with G_1, which gives a factor of I under the

[3]Note that in general, the permutation of the tensor product indices - denoted by primes in eq. (5) - must not be messed up with the permutation of spin states. Only in the case at hand, where 4 different spin states in 4 different tensor factors are considered, there is no difference.

exchange $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ or $\frac{1}{\sqrt{2}}(J+K)$ under $(1 \leftrightarrow 2)$, whereas a 2-fermion conglomerate in a $A_2 = c_1c'_2 - c_2c'_1$ configuration responds with -1 (i.e. antisymmetric) to the exchange of $(1 \leftrightarrow 2)$. See table 1, where the behavior of tetrons and fermions is compared. The fact that tetrons behave more complicated under transpositions $(i \leftrightarrow j)$, has to do with the fact that transpositions in S_4 correspond to relatively complicated space transformations in T_d.

We therefore conclude that tetrons follow their own statistics which is neither bosonic nor fermionic, and assert, that a sort of 'tetron spin statistics theorem' holds, which allows only bound states in which all tetron flavors are different. This then explains the selection rule/exclusion principle proposed in ref. [2] and mentioned after eq. (1).

To prove the assertion I can offer the following argument: remember that the Pauli principle for fermions demands antisymmetry (A_2 behavior) of a compound wave function under the exchange of all (spin and flavor) indices. In the case of tetrons one analogously needs a G_1 behavior of the compound wave function under the simultaneous permutations of all permutation indices (i,j,k,l and a,b,c,d), e.g. $a_i b'_j c''_k d'''_l \rightarrow b_j a'_i c''_k d'''_l$ for $(1 \leftrightarrow 2) \in S_4$. Such a behavior could not be obtained, if two or more flavor indices were identical.

4 Conclusions

It is certainly true that the phenomenological observation of 24 quarks and leptons and their interactions suggest a permutation principle only on the level of *inner* symmetries. However due to the problems which arise in connection with spin and statistics one is naturally lead to consider the possibility that inner and outer permutation behavior may be intertwined and that

FERMIONS	TETRONS
compound states:	
boson from 2 fermions: complex tensor product $A_2 = c_1 c_2' - c_2 c_1'$ bosonic behavior under rotations	fermion from 4 tetrons: quasi-complex, quaternion tensor product $G_1 = g(1,2,3,4) + Ig(2,1,4,3) + J...$ $= a_1 b_2' c_3'' d_4''' + I a_2 b_1' c_4'' d_3''' + ...$ fermionic behavior under rotations $G_1 \to (\alpha + J\beta) G_1$
permutation behavior/statistics:	
-1 under $(1 \leftrightarrow 2)$	a factor I under $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ a factor $\frac{1}{\sqrt{2}}(J+K)$ under $(1 \leftrightarrow 2)$ etc

Table 1: Comparison between the known fermion behavior and the anticipated tetron behavior.

this can be used to understand the spin-$\frac{1}{2}$ nature of quarks and leptons.

If the tetron approach has some meaning it is possible that besides G_1 also the two other half-integer spin representations of $\tilde{S}_4$ (H and G_2) play a role in nature, or in other words, that particles with spin $\frac{3}{2}$ and $\frac{5}{2}$ may appear at higher energy levels.

References

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