LITP-4/2009

TETRON MODEL BUILDING

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Abstract

Spin models are considered on a discretized inner symmetry space with tetrahedral symmetry as possible dynamical schemes for the tetron model. Parity violation, which corresponds to a change of sign for odd permutations, is shown to dictate the form of the Hamiltonian. It is further argued that such spin models can be obtained from more fundamental principles by considering a (7+1)-dimensional spacetime with octonion multiplication.

1 Introduction

Particle physics phenomena can be described, for example, by the left-right symmetric Standard Model with gauge group $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ [1] and 24 left-handed and 24 right-handed fermion fields which including antiparticles amounts to 96 degrees of freedom, i.e. this model has right handed neutrinos as well as righthanded weak interactions.

In recent papers [2, 3, 4] it was shown that there is a natural one-to-one correspondence between the quarks and leptons and the elements of the permutation group S_4, as made explicit in table 1 and natural in the sense that the color, isospin and family structure correspond to the K, Z_2 and Z_3 subgroups of S_4, [1] where Z_n is the cyclic group of n elements and K is the so-called Kleinsche Vierergruppe which consists of the 3 even permutations $\overline{2143}$, $\overline{3412}$, $\overline{4321}$, where 2 pairs of numbers are interchanged, plus the identity.

In other words, S_4 is a semi-direct product $S_4 = K \diamond Z_3 \diamond Z_2$ where the Z_3 factor is the family symmetry and Z_2 and K can be considered to be the 'germs' of weak isospin and color symmetry (cf. [3]). Furthermore it does not only describe quarks and leptons (table 1) but also leads to a new ordering scheme for the Standard Model gauge bosons, cf. ref. [2].

In refs. [2, 3, 4] a constituent picture was suggested where quarks and leptons are assumed to be built from 4 tetron 'flavors' a, b, c and d, whose interchanges generate the permutation group S_4. In the present paper I follow a somewhat different approach which relies on the fact that S_4 is also the symmetry group of a tetrahedral lattice or of a fluctuating S_4-permutation (quantum) lattice. In this approach the inner symmetry space is not continuous (with a continuous symmetry group) but has instead the discrete structure of a tetrahedral or S_4-permutation lattice, and the original dynamics is governed by some unknown lattice interaction instead of by four real tetron constituents.

The observed quarks and leptons can then be interpreted as excitations on

[1] S_4 is isomorphic to the rotational symmetry group of a regular tetrahedron and, up to a parity factor, the symmetry group of a 3-dimensional cube and octahedron.

	...1234... family 1	...1423... family 2	...1243... family 3
	τ, $b_{1,2,3}$	μ, $s_{1,2,3}$	e, $d_{1,2,3}$
ν	$\overline{1234}(id)$	$\overline{2314}$	$\overline{3124}$
u_1	$\overline{2143}(k_1)$	$\overline{3241}$	$\overline{1342}$
u_2	$\overline{3412}(k_2)$	$\overline{1423}$	$\overline{2431}$
u_3	$\overline{4321}(k_3)$	$\overline{4132}$	$\overline{4213}$
	ν_τ, $t_{1,2,3}$	ν_μ, $c_{1,2,3}$	ν_e, $u_{1,2,3}$
l	$\overline{3214}(1 \leftrightarrow 3)$	$\overline{1324}(2 \leftrightarrow 3)$	$\overline{2134}(1 \leftrightarrow 2)$
d_1	$\overline{2341}$	$\overline{3142}$	$\overline{1243}(3 \leftrightarrow 4)$
d_2	$\overline{1432}(2 \leftrightarrow 4)$	$\overline{2413}$	$\overline{3421}$
d_3	$\overline{4123}$	$\overline{4231}(1 \leftrightarrow 4)$	$\overline{4312}$

Table 1: List of elements of S_4 ordered in 3 fermion families. k_i denote the elements of K and $(a \leftrightarrow b)$ a simple permutation where a and b are interchanged. Permutations with a 4 at the last position form a S_3 subgroup of S_4 and may be thought of giving the set of lepton states. It should be noted that this is only a heuristic assignment. Actually one has to consider linear combinations of permutation states as discussed in section 2.

this lattice and characterized by representations of the lattice symmetry group S_4, i.e. by $A_1 + A_2 + 2E + 3T_1 + 3T_2$, just as in the 'classical' tetron model [2, 3, 4].

The lattice ansatz also naturally explains the selection rule mentioned in ref. [2] that all physical states must be permutation states: just because the lattice excitations must transform under representations of S_4.

In the following I will make the additional assumption that not only the inner symmetry is discrete but that physical space is a lattice, too. This line of thought takes up an old dream that field theoretical UV-infinities and renormalization problems can eventually be avoided by considering a fundamental theory living on a discretized instead of a continuous space. The average lattice spacing would be typically of the order of the Planck scale with the extension of the excitations somewhat larger. The reason for this assumption is that although theories with a discrete inner symmetry over a continuous base manifold have been examined [8] they seem to me rather artificial because they usually lead to domain walls and other discontinuities.

Quantum theory dictates that there is an uncertainty in the position of the lattice points. Therefore instead of a fixed spatial lattice one should assume that the lattice points are not a priori fixed but may be fluctuating, with the fluctuations following some (quantum) stochastic process [9]. When working in a semiclassical approximation one may neglect these fluctuations and consider a fixed lattice with tetrahedral or cubic symmetry.

There is some relation of this idea to other models which involve a fundamental length scale, like quantum foam models, which however assume gravity to play the central role in producing the new length scale, while in the tetron model gravitational interactions and cosmological phenomena appear only as byproducts of the spin lattice interactions [10].

In the present paper, dynamical models based on such lattices will be considered. They are typically spin models or fermionic lattice models. Variants of such models will be presented in the next sections: in section 2 a simple spin model on a 3-dimensional lattice will be discussed that describes the tetron phenomenology. In section 3 we follow the idea that the spatial and inner symmetry lattices can be unified to a larger lattice. One intriguing possibility is a 7-dimensional lattice involving octonions. This will be discussed in section 3, while in section 4 we come back to 3 dimensions with an octonion inspired model on a face centered cubic lattice.

2 Single-S Model

Starting with a spatial lattice, the most straightforward idea is to consider (inner symmetry) spin models. Spin models have been considered in statistical and solid state physics for a long time, and they have been used to describe magnetism and magnetic excitations as well as many other phenomena.

We start with a fixed cubic or tetrahedral 3-dimensional spatial lattice as discussed at the end of the last section, with a 'spin vector' $\vec{S}$ sitting on each lattice site.

In order to obtain the spectrum of the discrete S_4 group, one has to assume that the spin vectors take values in an at least 3-dimensional inner symmetry lattice. The dimension d_{in} of the inner symmetry lattice is restricted to be ≥ 3 because it is required to have a tetrahedral symmetry. The most straightforward choice is $d_{in} = 3$ but we shall also consider an example with $d_{in} = 4$.

What kind of Hamiltonian to choose? This is a very delicate question because the Hamiltonian eventually has to generate the full Standard Model phenomenology.

One essential requirement is that parity violation of the weak interactions should be described correctly. Usually the explanation of parity violation in subquark models is a real challenge. In the framework of the tetron idea the situation is somewhat simpler. The point is that in the tetron framework, as can be seen in table 1, weak isospin transformations are related to odd permutations. Therefore the Hamiltonian should transform non-trivially (i.e. antisymmetric) under odd permutations (of inner symmetry points, not of the base points). Furthermore, odd permutations of the inner symmetry group should be related to helicity/parity transformations in the sense that left handed transitions are energetically favoured as compared to righthanded ones.

A simple Heisenberg like spin model Hamiltonian fails to fulfil this requirement. However, there are two reasonable alternatives:

- on an inner 3-dimensional tetrahedral lattice there is the triple product

$$H = J \sum_t (\vec{S}_4 - \vec{S}_1)[(\vec{S}_3 - \vec{S}_1) \times (\vec{S}_2 - \vec{S}_1)] \tag{1}$$

  where J is the coupling strenght the sum runs over all tetrahedral plaquettes $t = 1, 2, 3, 4$ of the lattice and S_i is the value of spin vector on site i.

- on an inner 4-dimensional tetrahedral lattice there is the antisymmetric combination

$$H = J \sum_t \epsilon_{abcd} S_1^a S_2^b S_3^c S_4^d \tag{2}$$

  where S_i^a are the four components of the vector $\vec{S}_i$ sitting on site i.

In both cases the interaction is antisymmetric under odd permutations in the base lattice as well as in the inner symmetry lattice. This is precisely what is needed to describe the parity violation arising in particle physics.

A disadvantage of these Hamiltonians and of the use of spin models in general: it is not straightforward to construct fermionic excitation states. Usually, when one looks for fermionic excitations in spin models, one tries to write the spin vector as $\vec{S} = \psi^+ \vec{\sigma} \psi$ where ψ is a fermion on a given site and $\vec{\sigma}$ is the triplet of Pauli matrices. In that case one would have an effective 6-fermion interaction as the basic (effective) dynamical theory, which clearly calls for some more fundamental explanation.

Therefore I also consider a third scenario, in which not (locally bosonic) spin vectors but fermions sit on the lattice sites:

- on an 3-dimensional tetrahedral lattice consider the antisymmetric combination

$$H = J \sum_t \epsilon_{abcd} Q_1^a Q_2^b Q_3^c Q_4^d \tag{3}$$

where Q_i^a are the real components of a fundamental quaternion field sitting on site i. In other words, the interaction is the same as eq. (2) with the 'spin vectors' replaced by quaternion spinors.

This interaction is non-local in the sense that the fermion operators on different sites do not commute. Furthermore, one has to take care of the well-known problem of fermion doubling.

All the suggested Hamiltonians take care that the energy contribution to the partition function becomes negative, if an odd transformation is applied to the points of the base tetrahedron. This odd permutation is a reflection (proper rotation times parity transformation) both in the inner and in the spatial lattice. A combination of an odd and an even fermion corresponds to an odd weak vector boson (even×odd=odd).

3 Octonion Model

It would be nice to unify spatial and inner symmetry to one big lattice with a universal lattice constant. In the case of a 4-dimensional inner symmetry lattice eq. (2) this leads us directly to consider a spatial lattice on a (7+1)-dimensional space time $R(7,1)$ with symmetry SO(7,1).[2]

It is an old dream that inner symmetries may be obtained by extending ordinary 3-dimensional space to higher dimensions, and in particular to 7 dimensions, because a division algebra with a corresponding spinor structure can be defined there, namely the nonassociative and noncommutative algebra of octonions [5, 6, 7], which is related to SO(7) just as the non-commutative division algebra of quaternions is related to SO(3). More precisely, the group of unit octonions can be identified with the covering group Spin(7) of SO(7) just as the group of unit quaternions defines the covering group SU(2) of SO(3). Any evidence for an SO(4) inner symmetry in particle physics would long since have lead to physical models based on octonions with a 'compactification' $SO(7,1) \rightarrow SO(4) \times SO(3,1)$ that leads to

[2] In this paper I ignore gravity and consider all spaces to be flat.

a (3+1)-dimensional spacetime $R_{sp}(3,1)$ corresponding to a trivial fibration $R(7,1) \to R(4)_{in} \oplus R(3,1)_{sp}$ with fibers $R_{in}(4)$ (where *sp* and *in* stands for spatial and inner, respectively).

In the tetron model we do not have SO(4) but the discrete inner symmetry S_4. This may be considered a subgroup of SO(4), and this fact will now be used to suggest a tetron model based on a discretized 7-dimensional space, i.e. a lattice in $R(7)$. Concerning the 'compactification' of such a lattice we encounter a situation which is depicted graphically in figure 1 for the corresponding fibration $R(3) \to R(2)_{in} \oplus R(1)_{sp}$. Namely, on $R(4)_{in}$ there is a lattice with a natural S_4^{in} symmetry. This lattice can be explicitly constructed in the following way: We ignore quantum fluctuations of lattice points and make the semiclassical approximation of a fixed 7-dimensional lattice with a symmetry group which contains $S_4^{in} \times S_4^{sp}$. It is straightforward to define this lattice as the span of the regular 7-simplex in $R(7)$, i.e. to have it spanned by 8 linear independent unit vectors P_{1-8} regularly distributed on the 6-sphere in $R(7)$. The first four of these points P_{1-4} are assumed to span a regular tetrahedral lattice with symmetry group S_4^{sp} in what is assumed to be ordinary space. They can therefore be given in terms of quaternions I, J and $K = IJ$.

$$
\begin{aligned}
P_1 &= (-1,-1,-1) = -I - J - K \\
P_2 &= (-1,+1,+1) = -I + J + K \\
P_3 &= (+1,-1,+1) = I - J + K \\
P_4 &= (+1,+1,-1) = I + J - K
\end{aligned}
\tag{4}
$$

where as usual in quaternion construction I, J and K are taken to form an orthogonal basis of $R(3)$ (just as the octionion basis I,J,K,L,IL,JL and KL will be used form an orthogonal basis of $R(7)$). The rest of the points P_{5-8} span another tetrahedral lattice with symmetry S_4^{in} in another $R(3)$ within $R(7)$, which forms a tower of tetrahedral lattices in the 7th dimension. The situation is depicted graphically in figure 1 where instead of the reduction $R(7) \to R(3)_{in} \oplus R(3)_{sp}$ I have drawn $R(3) \to R^1_{in} \oplus R^1_{sp}$ and instead of symmetry group $S_4^{in} \times S_4^{sp}$ one has $S_2^{in} \times S_2^{sp}$ (the reflections of a line at the origin).

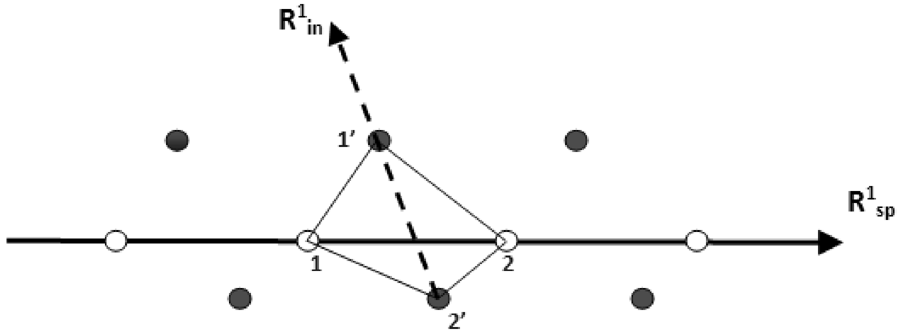

Figure 1: One dimensional spin chain from a three dimensional tetrahedral lattice in $R(3) \rightarrow R^1_{in} \oplus R^1_{sp}$ as visualization for $R(7) \rightarrow R(3)_{in} \oplus R(3)_{sp}$. The depicted lattice is assumed to possess a tetrahedral (S_4) symmetry with $S_4 \rightarrow S_2^{in} \times S_2^{sp}$. Transitions $1 \leftrightarrow 2$ correspond to spatial S_2 transformations, transitions $1' \leftrightarrow 2'$ to inner S_2 transformations.

Having set the geometrical framework one can now fix the fundamental field theoretical object to be an octonion sitting on each lattice site and taking values in a 8-dimensional lattice which corresponds to the span of the projective symmetry group of the original lattice. After the compactification this fundamental object will transform as (G_1^{in}, G_1^{sp}) under the lattice symmetry group $S_4^{in} \times S_4^{sp}$, where G_1 is the 2-dimensional spinor representation of the covering group of S_4. More precisely, the octonion can be written as

$$\begin{aligned} O &= Q_1 + LQ_2 \\ &= C_1 + JC_2 + L(C_3 + JC_4) \\ &= R_1 + IR_2 + J(R_3 + JR_4) + L(R_5 + IR_6 + J(R_7 + JR_8)) \end{aligned} \quad (5)$$

with quaternions Q_i, reals R_j and complex C_k. Q_1 and Q_2 give the representation G_1^{in} and G_1^{sp}, respectively.

As for the dynamics I suggest one of the following Hamiltonians:

- $$H = J \sum_t \epsilon_{abcd} C_1^a C_2^b C_3^c C_4^d + c.c. \tag{6}$$

  where C_i^a are the complex components of the fundamental octonion O_i sitting on site i as defined in eq. (5).

- $$H = J \sum_t (\vec{S}_4 - \vec{S}_1)[(\vec{S}_3 - \vec{S}_1) \times (\vec{S}_2 - \vec{S}_1)] \tag{7}$$

  where S_i is a 7-dimensional spin-vector S sitting on site i and a triple product can be defined in 7-dimensions.

4 FCC Lattice as an Octonion inspired Model

In this section I will shortly discuss a lattice model which works in 3 dimensions but is inspired or even induced by the 7 dimensional octonion model above.

I will assume that space has the structure of an fcc cubic lattice (like the NaCl crystal), i.e. there are 2 tetrahedral sublattices called Na and Cl with 2 types of inner spin vectors $\vec{S}$ and $\vec{T}$ sitting on the Na and Cl edges of the crystal.

Parity transformations correspond to the exchange of $\vec{S}$ and $\vec{T}$, while weak isospin is again related to odd permuations in the Na and Cl tetrahedral sublattices.

In contrast to the single-S model of section 2, one *does not* need to assume a *discrete* inner symmetry structure for $\vec{S}$ and $\vec{T}$, *because the S_4 inner symmetry which leads to quarks and leptons arises from relative rotations of the S and T sublattices.*

5 Conclusions

In conclusion, in the present paper lattice spin models have been discussed as possible dynamical schemes for the implementation of the tetron idea. Possible Hamiltonians have been presented, while the calculation of partition functions and expectation values for the excitation states is postponed to a forthcoming publication.

It is probable that the presented models are only effective descriptions of a more fundamental theory yet to be developed. However, presently there is no indication that this fundamental theory will have anything to do with the nowadays popular string or brane like structures. On the contrary, the appearance of S_4 symmetric states points to discrete structures at small distances and that the superstring ansatz is not opportune to describe natural phenomena.

Even more: the tetron idea is not only in opposition to string theories. It also reduces the celebrated gauge theories and the $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ Standard model gauge symmetry to what they are: a nice rather logical theoretical framework which however holds true only on a certain level of matter (the TeV energy range).

In fact, such a situation is not unusual in the development of science. It is well known from the macroscopic world as well as from nano physics, molecular and atomic physics that when going to a lower level of matter one has to give up the full understanding of some emergent phenomena known from the higher levels. In the present case we have given up continuus Lorentz invariance (which is restaurated at low energies) in favor of a fluctuating quantum lattice picture. Furthermore we consider the appearance of gauge symmetries as collective emergent effects.

It is possible that eventually the underlying structure (e.g. involving octonions as in section 3) turns out to be in some sense supersymmetric. However at the present stage I consider this option far from being compelling.

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