12/2008

Development of the Tetron Model

Bodo Lampe
e-mail: Lampe.Bodo@web.de

Abstract

The main features of the tetron model of elementary particles are discussed in the light of recent developments, in particular the formation of strong and electroweak vector bosons and a microscopic understanding of how the observed tetrahedral symmetry of the fermion spectrum may arise.

1 Prologue

In the left-right symmetric standard model with gauge group $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ [1] there are 24 left-handed and 24 right-handed fermion fields which including antiparticles amounts to 96 degrees of freedom, i.e. this model has right handed neutrinos as well as righthanded weak interactions.

In a recent paper [2] a new ordering scheme for the observed spectrum of quarks and leptons was presented, which relies on the structure of the group of permutations S_4 of four objects, and a mechanism was proposed, how 'germs' of the Standard Model interactions might be buried in this symmetry. In the following I want to extend this analysis in several directions. First, I will show that it is possible to embed the discrete S_4-symmetry in a larger continuous symmetry group. Afterwards, we shall see how the appearance of gauge bosons can be understood as well as obtain some hints about how the underlying microscopic structure may look like.

The permutation group S_4 [3] consists of 5 classes with altogether 24 elements $\sigma = \overline{abcd}$ where $a, b, c, d \in \{1, 2, 3, 4\}$. It has 5 representations A_1, A_2, E, T_1 and T_2 of dimensions 1, 1, 2, 3 and 3 and is isomorphic to the symmetry group T_d of a regular tetrahedron (and also to the subgroup O of proper rotations of the symmetry group O_h of a cube), cf table 1. The observed fermion symmetry will therefore be synonymously called T_d or S_4 in the following, depending on whether a geometrical or an algebraic viewpoint is taken.

An important subgroup of S_4 is A_4, the group of even permutations, which is sometimes called the 'symmetric group' and will be relevant in the discussion of gauge bosons in section 5. A_4 has 3 representations A, E and T of dimensions 1, 2 and 3 and is isomorphic to the symmetry group of proper

	S_4	T_d	O
I	$\overline{1234}(id)$	identity-rotation	identity-rotation
$3C_2$	$\overline{2143}$ $\overline{3412}$ $\overline{4321}$	rotations by π about the coordinate axes	rotation by π about the coordinate axes
$8C_3$	$\overline{2314}$ $\overline{3124}$ $\overline{3241}$ $\overline{1342}$ $\overline{1423}$ $\overline{2431}$ $\overline{4132}$ $\overline{4213}$	rotations by $\frac{2}{3}\pi$ about diagonals of the cube	rotations by $\frac{2}{3}\pi$ about diagonals of the cube
$6C_4$	6 transpositions $(i \leftrightarrow j)$ like $(1 \leftrightarrow 2) = \overline{2134}$	6 reflections on planes through the center and two edges i and j	rotations by $\pm\frac{1}{2}\pi$ about the coordinate axes
$6C_2'$	$\overline{2341}$ $\overline{3142}$ $\overline{2413}$ $\overline{3421}$ $\overline{4123}$ $\overline{4312}$	6 rotoreflections by $\frac{1}{2}\pi$	rotations by π about axes parallel to the 6 face diagonals

Table 1: Classes I, C_2, C_3, C_4 and C_2' of the groups S_4, T_d and O making their isomorphy explicit. Classes I, C_2 and C_3 form the 12-element subgroup A_4 of even permutations, which will be important in our analysis of vector bosons in section 5. The notation C_4 and C_2' is normally used only for rotations in O, whereas the classes of reflections in T_d are usually called σ and S_4 in the literature.

rotations of a regular tetrahedron.

The starting point of ref. [2] was the observation that there is a natural one-to-one correspondence between the fermion states and the elements of S_4. This feature is made explicit in table 2 where the elements of S_4 are associated to the fermions.

I use the term 'natural' because the color, isospin and family structure of fermions corresponds to K, Z_2 and Z_3 subgroups of S_4, where Z_n is the

	...1234... family 1	...1423... family 2	...1243... family 3
	τ, $b_{1,2,3}$	μ, $s_{1,2,3}$	e, $d_{1,2,3}$
ν	$\overline{1234}(id)$	$\overline{2314}$	$\overline{3124}$
u_1	$\overline{2143}(k_1)$	$\overline{3241}$	$\overline{1342}$
u_2	$\overline{3412}(k_2)$	$\overline{1423}$	$\overline{2431}$
u_3	$\overline{4321}(k_3)$	$\overline{4132}$	$\overline{4213}$
	ν_τ, $t_{1,2,3}$	ν_μ, $c_{1,2,3}$	ν_e, $u_{1,2,3}$
l	$\overline{3214}(1 \leftrightarrow 3)$	$\overline{1324}(2 \leftrightarrow 3)$	$\overline{2134}(1 \leftrightarrow 2)$
d_1	$\overline{2341}$	$\overline{3142}$	$\overline{1243}(3 \leftrightarrow 4)$
d_2	$\overline{1432}(2 \leftrightarrow 4)$	$\overline{2413}$	$\overline{3421}$
d_3	$\overline{4123}$	$\overline{4231}(1 \leftrightarrow 4)$	$\overline{4312}$

Table 2: List of elements of S_4 ordered in 3 families. k_i denote the elements of K and $(a \leftrightarrow b)$ a simple permutation where a and b are interchanged. Permutations with a 4 at the last position form a S_3 subgroup of S_4 and may be thought of giving the set of lepton states. It should be noted that this is only a heuristic assignment. Actually one has to consider linear combinations of permutation states as discussed in section 2.

(abelian) symmetric group of n elements and K is the so-called Kleinsche Vierergruppe which consists of the 3 even permutations $\overline{2143}$, $\overline{3412}$, $\overline{4321}$, where 2 pairs of numbers are interchanged (class C_2), plus the identity. In fact, S_4 is a semi-direct product $S_4 = K \diamond Z_3 \diamond Z_2$ where the Z_3 factor is the family symmetry and Z_2 and K can be considered to be the 'germs' of weak isospin and color (cf [2] and section 5). At low energies this product cannot be distinguished from the direct product $K \times Z_3 \times Z_2$ but has the advantage of being a simple group and having a rich geometric and group theoretical interpretation and will also lead to a new ordering scheme for the Standard Model vector bosons in section 5.

If one wants to include antiparticles and the spin of the fermions in this analysis, one can do the following: on the compound level the situation seems very simple. Spin and antiparticles each double the degrees of freedom, so that one has the structure of table 2 for f_L, f_R, $\bar{f}_L$ and $\bar{f}_R$ separately. This is enough, as long as one continues to consider quarks and leptons as pointlike objects, and asks questions like how under the assumption of the S_4 symmetry vector boson formation can be interpreted (section 5), and as long as one keeps the (discrete) inner and spatial symmetries completely separate - but it would not suffice any more, as soon as one would consider the possibility of compositeness and a spatial extension of the observed fermions, in particular in the form of a micro-geometric tetrahedral substructure cf. sect. 6 and ref. [2].

In that case the situation becomes more difficult, because one needs a description of the spacetime behavior of the constituents and how they join together to form a fermion. Ultimately, one would like to have a fully relativistic understanding of the fermion compound states. As a first step towards this goal in section 6 a nonrelativistic approach will be presented which relies on the representation G_1 of the covering group $\tilde{S}_4$ [6]. As will be shown, this accounts for an additional 'spin factor' in the fermion wave function which must be due the tetron spin degrees of freedom.

In effect, one has two functions f_σ^+ and f_σ^- for each flavor, where the spin averaged wave function is given by the sum

$$f_\sigma = f_\sigma^+ + f_\sigma^- \tag{1}$$

whereas the spin content is contained in the difference $f_\sigma^+ - f_\sigma^-$, so that including the spin degrees of freedom one has now 48 wave functions instead of the 24 given in table 1.

One may visualize this approach by a geometrical picture, where one has a cube which contains two tetrahedra (one for particles and the other one for antiparticles) which transform into each other by a CP-transformation so that for example in the process of vector boson formation $\bar{F}\gamma_\mu f$ the fermion f, which spreads over the first tetrahedron, and antifermion $\bar{F}$, which spreads over the other, join together to form a cube.

It should be noted that even if one rejects the constituent picture and/or the geometric intuition it is possible to give a meaning to the tetrahedra describing f_L and f_R and being connected by parity. Namely, in the SU(4) model which will be introduced in section 4, they do not live in physical space but exist as weight diagrams of the fundamental SU(4) representation. If one follows such an approach a correct relativistic treatment can be maintained without any difficulty.

2 The Use of Symmetry adapted Wave Functions and the Origin of strong and electroweak Charges

In [2] a sort of seesaw mechanism was derived which is able to accomodate all observed hierarchies in the quark and lepton masses. This mechanism relies on the introduction of S_4 symmetry functions to describe fermion fields, where the given Dirac fields of quarks and leptons are written as **symmetry adapted linear combinations** of more fundamental fields ψ_σ, $\sigma \in S_4$.

The linear coefficients are essentially given by the A_1, A_2, E, T_1 and T_2 representation matrices of S_4. This is due to the group theoretic theorem that from an arbitrary function $f(x)$ orthonormal sets of symmetry functions of a discrete group G can be obtained as

$$f_{ij} = \frac{dim(D)}{|G|} \sum_{g \in G} D_{ij}(g) f(g^{-1}x) \tag{2}$$

where D is any representation of G. (In general this will yield dim(D) sets of dim(D) orthonormal symmetry functions corresponding to the representation D.) Therefore to obtain the symmetry adapted functions one just has to take as linear coefficients the appropriate representation matrix entries D_{ij} which are well known in the realm of finite symmetry groups and for convenience given in tables 3 and 4 [5]. The resulting functions were already given in ref. [2].

In order to explain the observed parity violation of the weak *and* the $V - A$ structure of the strong interaction it was suggested [2] that the two tetrahedra describing fermions and antifermions are intertwined in the following sense: field components ψ_g corresponding to even permutations $g \in S_4$ live on one

		x	y	z	xyz	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}z$	$x\bar{y}\bar{z}$	xyz	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}z$	$x\bar{y}\bar{z}$
	1	C_2	C_2	C_2	C_8	C_8	C_8	C_8	C_8	C_8	C_8	C_8
A_1	1	1	1	1	1	1	1	1	1	1	1	1
A_2	1	1	1	1	1	1	1	1	1	1	1	1
$(E)_{11}$	1	1	1	1	c	c	c	c	c	c	c	c
$(E)_{21}$	0	0	0	0	s	s	s	s	-s	-s	-s	-s
$(E)_{12}$	0	0	0	0	-s	-s	-s	-s	s	s	s	s
$(E)_{22}$	1	1	1	1	c	c	c	c	c	c	c	c
$(T_1)_{11}$	1	1	-1	-1	0	0	0	0	0	0	0	0
$(T_1)_{21}$	0	0	0	0	1	-1	1	-1	0	0	0	0
$(T_1)_{31}$	0	0	0	0	0	0	0	0	1	-1	-1	1
$(T_1)_{12}$	0	0	0	0	0	0	0	0	1	1	-1	-1
$(T_1)_{22}$	1	-1	1	-1	0	0	0	0	0	0	0	0
$(T_1)_{32}$	0	0	0	0	1	-1	-1	1	0	0	0	0
$(T_1)_{13}$	0	0	0	0	1	1	-1	-1	0	0	0	0
$(T_1)_{23}$	0	0	0	0	0	0	0	0	1	-1	1	-1
$(T_1)_{33}$	1	-1	-1	1	0	0	0	0	0	0	0	0
$(T_2)_{11}$	1	1	-1	-1	0	0	0	0	0	0	0	0
$(T_2)_{21}$	0	0	0	0	1	-1	1	-1	0	0	0	0
$(T_2)_{31}$	0	0	0	0	0	0	0	0	1	-1	-1	1
$(T_2)_{12}$	0	0	0	0	0	0	0	0	1	1	-1	-1
$(T_2)_{22}$	1	-1	1	-1	0	0	0	0	0	0	0	0
$(T_2)_{32}$	0	0	0	0	1	-1	-1	1	0	0	0	0
$(T_2)_{13}$	0	0	0	0	1	1	-1	-1	0	0	0	0
$(T_2)_{23}$	0	0	0	0	0	0	0	0	1	-1	1	-1
$(T_2)_{33}$	1	-1	-1	1	0	0	0	0	0	0	0	0

Table 3: Matrices for the irreducible representations of $S_4 = T_d$ fixing the coefficients of the symmetry adapted functions as given in [5]. I have used the abbreviation $c = cos(\frac{2}{3}\pi) = -\frac{1}{2}$ and $s = sin(\frac{2}{3}\pi) = \frac{\sqrt{3}}{2}$.

	$\bar{x}y$	xy	$\bar{x}z$	xz	$\bar{y}z$	yz	z	z	y	y	x	x
	σ	σ	σ	σ	σ	σ	S_4	S_4	S_4	S_4	S_4	S_4
A_1	1	1	1	1	1	1	1	1	1	1	1	1
A_2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$(E)_{11}$	1	1	c	c	c	c	1	1	c	c	c	c
$(E)_{21}$	0	0	s	s	-s	-s	0	0	s	s	-s	-s
$(E)_{12}$	0	0	s	s	-s	-s	0	0	s	s	-s	-s
$(E)_{22}$	-1	-1	-c	-c	-c	-c	-1	-1	-c	-c	-c	-c
$(T_1)_{11}$	0	0	0	0	-1	-1	0	0	0	0	1	1
$(T_1)_{21}$	-1	1	0	0	0	0	-1	1	0	0	0	0
$(T_1)_{31}$	0	0	-1	1	0	0	0	0	1	-1	0	0
$(T_1)_{12}$	-1	1	0	0	0	0	1	-1	0	0	0	0
$(T_1)_{22}$	0	0	-1	-1	0	0	0	0	1	1	0	0
$(T_1)_{32}$	0	0	0	0	-1	1	0	0	0	0	-1	1
$(T_1)_{13}$	0	0	-1	1	0	0	0	0	-1	1	0	0
$(T_1)_{23}$	0	0	0	0	-1	1	0	0	0	0	1	-1
$(T_1)_{33}$	-1	-1	0	0	0	0	1	1	0	0	0	0
$(T_2)_{11}$	0	0	0	0	1	1	0	0	0	0	-1	-1
$(T_2)_{21}$	1	-1	0	0	0	0	1	-1	0	0	0	0
$(T_2)_{31}$	0	0	1	-1	0	0	0	0	-1	1	0	0
$(T_2)_{12}$	1	-1	0	0	0	0	-1	1	0	0	0	0
$(T_2)_{22}$	0	0	1	1	0	0	0	0	-1	-1	0	0
$(T_2)_{32}$	0	0	0	0	1	-1	0	0	0	0	1	-1
$(T_2)_{13}$	0	0	1	-1	0	0	0	0	1	-1	0	0
$(T_2)_{23}$	0	0	0	0	1	-1	0	0	0	0	-1	1
$(T_2)_{33}$	1	1	0	0	0	0	-1	-1	0	0	0	0

Table 4: Continuation of table 3: representation matrices for the reflection operations in T_d. The symbols above the symmetry operations indicate their orientation relative to the axes.

tetrahedron, whereas components ψ_u corresponding to odd permutations $u \in S_4$ live on the other. In other words, the symmetry adapted functions for left handed fermions have the generic form $f_L = \psi_g + P\psi_u$ and those for the right handed $f_R = P\psi_g + \psi_u$. The point is that fermions of opposite isospin differ by an odd permutation (as is explicit from table 2), so that parity violation/conservation for weak bosons/gluons is obtained.[2]

Having made extensive use of symmetry adapted functions in various directions, it is time to discuss the legitimacy and drawbacks of such an approach, which have to do with the fact that one is combining fields with different Standard Model charges into linear combinations. As a consequence no definite strong and electroweak charges can be associated to single state componenents ψ_σ, $\sigma = \overline{abcd} \in S_4$, but only to the symmetry adapted linear combinations giving the quarks and leptons. In other words, such an approach can only be valid, if **the Standard Model charges arise as derived entities from secondary dynamical causes and are not really fundamental.** Fundamental are only the interactions behind the S_4-symmetry (resp. SU(4)-symmetry in section 4) or the superstrong forces between the possible constituents, whereas the Standard Model interactions of the fermions do not exist a priori but are just a consequence of the relative position of a, b, c and d in the permutations. In order to understand this more clearly it was suggested in [2] to introduce nondiagonal charge operators so that not the permutation fields ψ_σ but their symmetry combinations are eigenfunctions of the Standard Model charge operators - in much the same way as they are not eigenfunctions of the mass operator.

If one does not like this approach and wants to stick to the viewpoint that charge operators must be diagonal and have to be associated not to linear combinations of fields but to the fields ψ_σ themselves, one has to give up the symmetry adapted linear combinations. The only linear combinations which may then be used are Z_3-adapted functions, because they are not associated to any charges but to the family symmetry. In other words, since for example the 3 neutrinos, for which permutations of the first 3 indices are relevant (cf table 2), have identical Standard Model charges, one may use linear combinations of the form

$$\nu_e = \psi_{\overline{1234}} + \psi_{\overline{2314}} + \psi_{\overline{3124}} \tag{3}$$

$$\nu_\mu = \psi_{\overline{1234}} + \epsilon\psi_{\overline{2314}} + \epsilon^*\psi_{\overline{3124}} \tag{4}$$

$$\nu_\tau = \psi_{\overline{1234}} + \epsilon^*\psi_{\overline{2314}} + \epsilon\psi_{\overline{3124}} \tag{5}$$

and similarly for electron, muon and tau-lepton

$$e = \psi_{\overline{3214}} + \psi_{\overline{1324}} + \psi_{\overline{2134}} \tag{6}$$

$$\mu = \psi_{\overline{3214}} + \epsilon\psi_{\overline{1324}} + \epsilon^*\psi_{\overline{2134}} \tag{7}$$

$$\tau = \psi_{\overline{3214}} + \epsilon^*\psi_{\overline{1324}} + \epsilon\psi_{\overline{2134}} \tag{8}$$

These equations are easily understood because Z_3-symmetry combinations always have the generic form $f_0 + f_1 + f_2$, $f_0 + \epsilon f_1 + \epsilon^* f_2$ and $f_0 + \epsilon^* f_1 + \epsilon f_2$, where $\epsilon = exp(2\pi i/3)$.

Gauge bosons may be re-expressed using these combinations. For example one obtains for the leptonic part of the neutral weak W-boson

$$W_{3\mu} = \bar{e}\gamma_\mu e - \bar{\nu}_e\gamma_\mu\nu_e + \bar{\mu}\gamma_\mu\mu - \bar{\nu}_\mu\gamma_\mu\nu_\mu + \bar{\tau}\gamma_\mu\tau - \bar{\nu}_\tau\gamma_\mu\nu_\tau \tag{9}$$

$$= 3(\bar{\psi}_{\overline{1234}}\gamma_\mu\psi_{\overline{1234}} + \bar{\psi}_{\overline{2314}}\gamma_\mu\psi_{\overline{2314}} + \bar{\psi}_{\overline{3124}}\gamma_\mu\psi_{\overline{3124}} - \bar{\psi}_{\overline{3214}}\gamma_\mu\psi_{\overline{3214}} - \bar{\psi}_{\overline{1324}}\gamma_\mu\psi_{\overline{1324}} - \bar{\psi}_{\overline{2134}}\gamma_\mu\psi_{\overline{2134}}) \tag{10}$$

Note that eqs. (3)-(10) hold separately for left and right handed lepton and W fields.

3 The two main Problems

In the remainder of this work I will deal with the two fundamental problems, which have to be solved, if the tetron approach is to make sense:

- First to understand in a natural way the appearance of vector bosons as linear combinations of products of fermion fields. In particular the question why among the many fermion-antifermion products which can in principle be formed, precisely and only those corresponding to the Standard Model gauge groups arise. The idea which reduces the number of possible combinations and produces the Standard Model gauge bosons will be that when product states are formed from two fermions each with T_d- resp. O_h-symmetry a final state object appears, which again has a symmetry of (a subgroup of) T_d.

- Secondly what the origin of the tetrahedral symmetry may be. It is plausible although not compelling that the observed S_4-symmetry points to a substructure of quarks and leptons with four constituents (called tetrons). In this scenario the main question concerns the space time behavior of the tetrons, and in particular how the spin-$\frac{1}{2}$ nature of the observed fermions can be obtained. One possibility, which will be followed in section 6, is to give up continuous spatial rotation symmetry on the microscopic level and replace it by a discrete symmetry and then to consider Z_4-extensions of the tetrahedral group instead of the Z_2-extension defined by the covering group. This will involve the use of octonions and giving up complex in favor of quaternion quantum mechanics.

4 Discrete versus continuous inner Symmetry

I have repeatedly mentioned the argument of ref.[2] that S_4-symmetry transformations may serve as 'germs' for the gauge symmetries which in modern times are used to describe the strong and electroweak interactions.

Discrete symmetry as an ordering scheme for quarks and leptons and a possible source for their interactions? At this point particle physicists may feel a bit uneasy, because it can hardly be imagined that the rich and rather involved structure of the Standard Model gauge theories can be derived in a strict sense from a discrete symmetry structure.

Therefore, one may look for alternative ideas, and one possibility is that the appearant S_4-symmetry of quarks and leptons is part of a larger (continuous) symmetry group like SU(4) or Sp(4). In these groups the S_4-symmetry adapted functions naturally appear as part of the product states in $\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4}$, where $\mathbf{4}$ is the fundamental representation of SU(4), the representation space being spanned by 'tetron' states a, b, c and d, just like in the $SU(3)_{flavor}$ quark model the fundamental representation $\mathbf{3}$ is spanned by fields u, d and s. The point is that if one considers fourfold tensor products $\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4}$, among the corresponding 256 possible states one will automatically encounter the 24 linear combinations of product states $\psi_{\overline{abcd}} = a \times b \times c \times d$ and their permutations, or more precisely the symmetry adapted linear combinations thereof - just like in the $SU(3)_{flavor}$ quark model among the 27 baryonic states in $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ there are 6 linear combinations like for example $\Lambda^0 = \frac{1}{\sqrt{12}}[sdu - sud + usd - dsu + 2(uds - dus)]$ which can be interpreted as symmetry adapted functions of the permutation group S_3. This is not astonishing but has to do with the fact that $S_4(S_3)$ is a distinct particle symmetry of the product states in $\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4}$ ($\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$).

Since the fundamental representation of SU(4) can be geometrically visualized as a tetrahedron which lives in a 3-dimensional weight diagram spanned by the SU(4) generators $\lambda_{3,8,15}$, we arrive at more or less the same geometrical picture as described in section 1 for the discrete S_4-symmetry. Even the formation of vector bosons as compounds $\bar{F}\gamma_\mu f$ from two tetrahedral configurations, which can be transformed into another by CP and where a cube is formed in the combined weight diagram of particles and antiparticles, can be understood in this model.

There are 3 questions left open:

- how the Standard Model charges and interactions can arise from an SU(4) 'hyperflavor' interaction just by a permutation of constituents. This question will be tackled in section 5.
- how products of 4 constituents can make up for fermions with their spin-$\frac{1}{2}$ transformation properties under spatial rotations. This will be discussed in section 6 and further in a forthcoming publication [4].
- and finally why only 'distinct'-tetron states arise, whereas all the rest of the 256 product states (those where one of the tetrons appears at least twice) are not observed (or have a much higher mass).

As for the last problem I formulate the following **exclusion principle for tetrons**: quarks and leptons consist of 4 tetron states a,b,c,d. Only states where all tetrons are different are allowed. In order to include vector bosons and their treatment in section 5 one may extend this principle as follwows: for an arbitrary state to be physical the exclusion principle demands that it is part of a S_4 permutation multiplet.

Note that this is a weaker condition (i.e. allows more states) than for example the color singlet principle of $SU(3)_{color}$-QCD, which demands that among all $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ only the A_2 singlet function $\epsilon(i,j,k)q_i q_j q_k$ is allowed.

In conclusion one may say that one has two options which match the phenomenological fermion spectrum equally well: either one uses a continuous inner symmetry group like SU(4) *together with* an exclusion principle or one sticks to the discrete tetrahedral=permutation symmetry.

One can make the connection between these two approaches explicit by writing down the T_d-content of the relevant SU(4) representations. Namely, within the discrete approach the 24 fermion states can be classified according to the T_d representations A_1, A_2, E, T_1 and T_2, i.e. the 18 T_1- and T_2-functions are used to describe up- and down-type quarks degrees of freedom respectively, whereas the 6 A_1-, A_2- and E-functions are responsible for leptons. (This is just the use of the symmetry adapted functions discussed before and in [2].) On the other hand, in the continuous symmetry approach the 256 SU(4)-states of $\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4}$ may be decomposed according to

$$\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} = 3\times\mathbf{45(T_1)} + 3\times\mathbf{15(T_2)} + 2\times\mathbf{20(E)} + \mathbf{35(A_1)} + \mathbf{1(A_2)} \quad (11)$$

Here one finds in brackets, which kind of T_d symmetry functions are contained in the corresponding SU(4) representations. For example, there are three SU(4) representations of dimension 45 each containing a set of 3 T_1-functions, i.e. all in all the 9 functions used to describe the up-type quarks. More precisely, the 3 functions of the n-th T_1 in (11) are to describe the family triplet u_n, c_n and t_n, where n=1,2,3 is the color index. Similarly there are 3 sets of 3 T_2-functions in the 3 15-dimensional representations to describe the down-type quarks. Furthermore, A_1 and A_2 describes the electron and its neutrino, whereas one E-representation in (11) contains μ and τ and the other ν_μ and ν_τ.

It is an interesting observation that this way only particles of the same Standard Model charges (but belonging to different families) are put together in a SU(4) multiplet. The alternative would be to put quarks of different color into one SU(4) multiplet (like u_1, u_2 and u_3 into one **45**) and similarly for leptons of different isospin (e.g. μ and ν_μ into one **20**).

One should stress that the choice of SU(4) is not compelling. One could choose other groups which contain S_4 and its representations, like SO(4) or even SO(3). However, I consider the possibility, where four constituent tetrons span the basis of the fundamental complex representation the most promising.

It should further be noted that the fermion mass relations derived in [2] on the basis of the discrete T_d-symmetry can be rederived as SU(4) mass relations that are analogous to the mass relations for hadrons derived in the $SU(3)_{flavor}$ quark model.

5 Vector Boson Formation

In this section I will not make any assumptions about possible substructures of quarks and leptons, but will only use the appearant S_4-symmetry of their spectrum table 2. On the basis of this symmetry I want to show that the vector bosons of the left-right symmetric Standard Model can be ordered in a similar manner and according to the same principle as the fermions. The idea is that the tetrahedral (resp octahedral) symmetry of the quarks and leptons is more or less retained when the vector bosons are formed. More precisely, I shall assume that the vector boson states can be ordered according to the subgroup A_4 of S_4 (the so called symmetric group of even permutations). This reduces the a priori large number of possible fermion-antifermion interactions, because it means that whatever internal dynamical reordering takes place in the process of vector boson formation $\bar{F}\gamma_\mu f$ from two fermions F and f, the resulting state has to have A_4 symmetry. For example, the long discussion in ref.[2] of how to avoid leptoquarks is completely superfluous in this approach simply because within the A_4-symmetry with its 12 degrees of freedom there is no space for additional gauge bosons.

The two possible types of vector bosons $V_{\mu L} = \bar{F}_L \gamma_\mu f_L$ and $V_{\mu R} = \bar{F}_R \gamma_\mu f_R$ can be accounted for by including parity $P : V_{\mu L} \leftrightarrow V_{\mu R}$ so that one arrives at the so called pyritohedral symmetry $A_4 \times P$, a subgroup of the octahedral group O_h. Note that since the gauge bosons have spin 1, no covering group has to be considered. Note further that since I work in the relativistic limit (which I can do since S_4 and A_4 are just inner symmetries of pointlike particles) no vector boson spin-0 component appears.

In table 5 I present a heuristic ordering of the observed vector bosons according to the proposed A_4-symmetry. Phenomenologically, there are 8 gluons G_μ, one $(B-L)$-photon B_μ and 3 weak bosons $W_{1,2,3\mu}$. The argument of why only the weak bosons appear in a right- and a lefthanded version W_R and W_L, whereas for gluons and photon one has $G_{\mu L} = G_{\mu R}$ and $B_{\mu L} = B_{\mu R}$ can be taken over from ref [2].

This table, which may look miraculous at first sight, is not difficult to understand. For example, in [2] it was argued that the weak bosons $W_{1,2,3}$ arise naturally from the Kleinsche Vierergruppe K (the subgroup of A_4 formed by the classes I and $3C_2$) because it is isomorphic to $Z_2 \times Z_2$ where the two Z_2 factors stand for the germs of weak isospin of the fermion resp antifermion.

To go beyond such a heuristic understanding one should use symmetry adapted linear combinations of functions Ψ_σ, $\sigma \in A_4$ instead of the simple assignments of table 5. The linear coefficients could in principle be taken from

$B_\mu = \overline{1234}$	$G_{3\mu} = \overline{2314}$	$G_{8\mu} = \overline{3124}$
$W_{3\mu} = \overline{2143}$	$G_{1\mu} = \overline{3241}$	$G_{2\mu} = \overline{1342}$
$W_{1\mu} = \overline{3412}$	$G_{4\mu} = \overline{1423}$	$G_{5\mu} = \overline{2431}$
$W_{2\mu} = \overline{4321}$	$G_{6\mu} = \overline{4132}$	$G_{7\mu} = \overline{4213}$

Table 5: List of Standard Model vector bosons ordered heuristically according to their proposed A_4 symmetry. A_4 is composed of 3 classes I, $3C_2$, $8C_3$ (cf table 1) and the proposed ordering follows this line. Note that just as table 2 for fermions these are only preliminary assignments. Later we shall see, how to construct the correct vector bosons states in terms of symmetry adapted functions.

table 3 (dropping the contributions from improper rotations). However we shall instantly see how to construct them explicitly from fermion-antifermion bilinears in order to obtain the combinations relevant in particle physics.

Using S_4-Clebsch-Gordon coefficients for the fermion-antifermion tensor products [8], I want to show, that and how from the 24x24=512 possible fermion-antifermion-product states 12 are selected in order to describe the final states (the vector bosons). From the point of principle this is in fact no question: if the final states are to have A_4-symmetry then their number *must* boil down to 12. In practice these states can be explicitly constructed by evaluating fermion-antifermion products using the S_4-symmetry adapted functions for the fermions whose benefits and deficiencies have been discussed in section 2, also in connection with their appearance in the continuous SU(4) model in section 4, cf. eq. (11), projecting them to $A_4 \subset S_4$ and comparing the result with the observed vector boson spectrum.

Before I start I want to remind the reader that the 24 S_4-functions for fermions divide into 9 symmetry functions from T_1 used for the up-type-quarks, 9 functions from T_2 for the down-type-quarks and 6 functions from A_1, A_2 and E for the lepton degrees of freedom and that they all can be obtained from table 3. Clebsch-Gordon(CG) coefficients appear when one calculates tensor products of two representations D_1 and D_2 as direct sums

$$D_1 \otimes D_2 = D_3 \oplus ... \tag{12}$$

and wants to determine a set of symmetry functions for D_3 from symmetry functions f_1^i and f_2^j of D_1 and D_2. Namely they are given

$$f_3^k = \sqrt{dim(D_3)} \sum_{i,j} V(D_1, D_2, D_3, i, j, k) f_1^i f_2^j \tag{13}$$

where the sum runs over sets of symmetry functions that span the representation spaces, $i = 1, ..., dim(D_1)$ and $j = 1, ..., dim(D_2)$. Eq. (13) will be used as the defining equation for the normalization of the CG-coefficients. (In fact we are using so-called V-coefficients which have the advantage of being invariant under simultaneous permutations of representations and indices in their argument.)

Consider for example the product $T_1 \otimes T_1$. Since T_1 corresponds to the up-type quarks, the product $T_1 \otimes T_1$ will yield 9 up-quark bilinears. Within S_4 these can be decomposed according to

$$T_1 \otimes T_1 = A_1 \oplus E \oplus T_1 \oplus T_2 \tag{14}$$

Taking the 3 up-quark color components u_1, u_2 and u_3 as T_1-functions on the LHS and evaluating the corresponding Clebsch-Gordon coefficients leads to

- a representation of the $(B-L)$-photon as

$$B_\mu = \bar{u}_1 \gamma_\mu u_1 + \bar{u}_2 \gamma_\mu u_2 + \bar{u}_3 \gamma_\mu u_3 \tag{15}$$

This stems from the representation A_1 on the right hand side of eq. (14) and from the corresponding Clebsch-Gordon coefficient [8]

$$V(T_1, T_1, A_1; i, j, 1) = \frac{1}{\sqrt{3}}\delta_{ij} \tag{16}$$

- a representation of the gluon octet stemming from the remaining part $E \oplus T_1 \oplus T_2$ of the decomposition eq. (14). Namely, the CG-coefficients can be written in terms of the Gell-Man λ-matrices as

$$\begin{aligned}
V(T_1, T_1, T_1; i, j, k) &= \frac{1}{\sqrt{6}}\epsilon_{ijk} && (17)\\
&= \frac{i}{\sqrt{6}}\lambda_{7,5,2ij} \quad for \quad k = 1, 2, 3 && (18)\\
V(T_1, T_1, T_2; i, j, k) &= \frac{1}{\sqrt{6}}|\epsilon_{ijk}| && (19)\\
&= \frac{1}{\sqrt{6}}\lambda_{6,4,1ij} \quad for \quad k = 1, 2, 3 && (20)\\
V(T_1, T_1, E; i, j, 1) &= \frac{1}{2}\lambda_{8ij} && (21)\\
V(T_1, T_1, E; i, j, 2) &= \frac{1}{2}\lambda_{3ij} && (22)\\
& && (23)
\end{aligned}$$

Note that the difference in the coefficients $\frac{1}{2}$ of $V(T_1, T_1, E)$ and $\frac{1}{\sqrt{6}}$ of $V(T_1, T_1, T_{1,2})$ is an artefact of the normalization factor $\sqrt{dim(D_3)}$ in eq. (13). All in all we obtain

$$\begin{aligned}
G_{3\mu} &= \bar{u}_1\gamma_\mu u_1 - \bar{u}_2\gamma_\mu u_2 && (24)\\
G_{8\mu} &= \frac{1}{\sqrt{3}}(\bar{u}_1\gamma_\mu u_1 + \bar{u}_2\gamma_\mu u_2 - 2\bar{u}_3\gamma_\mu u_3) && (25)
\end{aligned}$$

and similarly for the other λ-matrices.

The fact that formally the same bilinear combinations are created as needed in $SU(3)_{color}$-QCD is no accident but has to do with the fact that $S_4 = T_d$ is a subgroup $T_d \subset SO(3) \subset SU(3)$. The result is therefore an elaboration on the claim formulated in [2] that the appearant tetrahedral symmetry of quarks and leptons is able to provide 'germs' of the Standard Model gauge interactions.

It should further be noted that there is no problem of antifermions being involved here, because on the S_4 level there is no difference in the treatment of fermion-fermion and fermion-antifermion bilinears, because the group tensor product states do not care whether they are formed with particles or antiparticles.

Nevertheless, one could have the suspicion of being cheated here in that one obtains complex fields from real representations of a discrete group. To be on the safe side, one may embed these considerations in the framework of the SU(4) model presented in section 4. In that model the physical vector bosons will be states in the representation $(\bar{\mathbf{4}} \otimes \bar{\mathbf{4}} \otimes \bar{\mathbf{4}} \otimes \bar{\mathbf{4}}) \otimes (\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4})$[1]. What is done in this section is to select the 12 physical vector bosons among the 4^8 states in that representation by applying the exclusion principle ('any physical particle must be a permutation state') proposed in section 4.

As a next step the results eqs. (14)-(25) have to be projected from S_4 to A_4 of the vector bosons. This can be done by symmetrization in the family (u,c,t) and the isospin (up,down) degrees of freedom. Doing that the gluons turn out all right, but the $(B-L)$-photon is still missing its lepton contributions.

The point is that A_4 has a 3-dimensional representation T (for which 9 sym-

[1]Such vector boson bound states correspond in a sense to a kind of technicolor theory. Technicolor models often predict a large value for the Peskin-Takeuchi S and T parameters [7] and non universal couplings for the third generation. However, since I am not in the stage of proposing a specific tetron dynamics, it may be possible that some elaborate dynamical scheme exists where these deviations are canceled.

metry functions are needed), a 2-dimensional representation E (with only 2 functions because it is separably degenerate) and the totally symmetric representation A[2]. Interpreted on this basis we obtain from the RHS of eq. (14):
i) the symmetry function for the totally symmetric representation A
ii) the two symmetry functions for the representation E
iii) 6 of the 9 T-functions (3 from T_1 and 3 from T_2).

The 3 missing T-functions, which will be used to describe the weak bosons, can be obtained, for example, from the product

$$E \otimes E = A_1 \oplus A_2 \oplus E \tag{26}$$

Namely, taking μ and ν_μ as basis functions for E on the LHS and evaluating the corresponding Clebsch-Gordon coefficients leads to

- a representation of the $(B-L)$-photon as $B_\mu = \bar{\nu}_\mu \gamma_\mu \nu_\mu + \bar{\mu} \gamma_\mu \mu$ which is due to the A_1-term in eq. (26) and, after symmetrization over the family index, gives in fact the missing lepton part of the quark-lepton symmetrized representation of B_μ.

- a representation of the weak boson triplet stemming from the remaining part $A_2 \oplus E$ of the decomposition eq. (26). Namely, the CG-coefficients $V(E, E, A_2)$ and $V(E, E, E)$ are given by

$$V(E, E, A_2; 1, 1, 1) = 0 \qquad V(E, E, A_2; 1, 2, 1) = \frac{1}{\sqrt{2}} \tag{27}$$

$$V(E, E, A_2; 2, 1, 1) = -\frac{1}{\sqrt{2}} \qquad V(E, E, A_2; 2, 2, 1) = 0 \tag{28}$$

[2]In the literature the represenations of A_4 are sometimes labelled as **1**, **1'**, **1"** and **3**, where **1**=A and **3**=T and **1'** and **1"** are two one-dimenensional representations to which the representation I have called E, can be effectively reduced by an appropriate base transformation (note this is only possible for A_4 but not for the E of S_4 and has to do with the fact how the Z3-subgroup is embedded in A_4.)

and

$$V(E,E,E;1,1,1) = -\frac{1}{2} \qquad V(E,E,E;1,2,1) = 0 \tag{29}$$
$$V(E,E,E;2,1,1) = 0 \qquad V(E,E,E;2,2,1) = \frac{1}{2} \tag{30}$$
$$V(E,E,E;1,1,2) = 0 \qquad V(E,E,E;1,2,2) = \frac{1}{2} \tag{31}$$
$$V(E,E,E;2,1,2) = \frac{1}{2} \qquad V(E,E,E;2,2,2) = 0 \tag{32}$$

leading to the combinations

$$W_1 = \bar{\mu}\gamma_\mu\nu_\mu + \bar{\nu}_\mu\gamma_\mu\mu \tag{33}$$
$$iW_2 = \bar{\mu}\gamma_\mu\nu_\mu - \bar{\nu}_\mu\gamma_\mu\mu \tag{34}$$
$$W_3 = \bar{\mu}\gamma_\mu\mu - \bar{\nu}_\mu\gamma_\mu\nu_\mu \tag{35}$$

Writing the CG-coefficients eqs. (27)-32) in terms of Pauli matrices σ

$$V(E,E,A_2;i,j,1) = \frac{i}{\sqrt{2}}\sigma_{2ij} \tag{36}$$
$$\frac{1}{\sqrt{2}}V(E,E,E;i,j,2) = \frac{1}{\sqrt{2}}\sigma_{1ij} \tag{37}$$
$$\frac{1}{\sqrt{2}}V(E,E,E;i,j,1) = \frac{1}{\sqrt{2}}\sigma_{3ij} \tag{38}$$

it becomes appearant that they are formally a $SU(2)_{weak}$ triplet. Since the T-representation of A_4 is the restriction of the triplet representation to A_4 considered as a subgroup of $SU(2)_{weak}$ they can be used as the set of missing symmetry functions for T.

As before the result eq. (33)-(35) has to be symmetrized in the family and the quark and lepton degrees of freedom.

Let me finish this section with the remark that predictions of vector boson mass ratios can in principle be made from the preceeding symmetry considerations in the same way as was done for the fermion mass spectrum in ref. [2]. In the A_4-symmetric limit all vector bosons are massless. When A_4 gets broken, one can make an ansatz similar to that for fermions in [2], namely using the fact that the electro-strong and weak vector bosons are contained in different representations of A_4 they get different masses. Thus one can naturally accommodate massless photons and gluons and massive weak bosons. The question how the observed large mass difference between the left and right handed weak bosons arise remains open at this point.

6 Octonions - a possible Solution to the Tetron Spin Problem

In the preceeding chapters the inner symmetries of the tetron model have been discussed on the basis of the known representations of the permutation group S_4 without much reference to a possible tetronic substructure and how the fermion spatial behavior of quarks and leptons may arise on a microscopic level.

In contrast, from now on I will explicitly assume that quarks and leptons are built from 4 tetron constituents. The main problem is then to construct the spin-$\frac{1}{2}$ behavior of the compound fermions from the spacetime properties of the tetrons.

I will mainly present the framework, in which this problem should be solved, and discuss a possible solution at the end, which is based on an octonion Z_4 extension of the rotation group SO(3).

I will consider spatial transformations only. The extension to Minkowski space will be worked out in a separate publication [4].

Let me start with a few well-known facts about half-integer spin: in a physical experiment one cannot distinguish between states which differ by a complex phase. Therefore, in addition to ordinary representations one may include projective, half-integer spin representations of the rotation group SO(3), and also of its $T_d = S_4$ subgroup[3]. These are true representations of the corresponding covering groups SU(2) and $\tilde{S}_4$, respectively.

To solve the tetron spin problem I suggest to give up two standard elements of quantum mechanics:

- Firstly, I will give up the principle of a complex in favor of an octonion quantum mechanics [13, 14]. One may then have octonion projective representations (e.g. of nontrivial Z_4-extensions) of the rotation group. Furthermore, in such a framework tetron states should be combined using octonion instead of complex tensor products.
- Secondly, I will give up the requirement of continuous rotation symmetry and assume that tetrons live and interact in microscopical dimensions, in which only permutation symmetry survives. The latter is much less restrictive than rotational SO(3), because the idea of rotation assumes concepts of angle and length, which may be obstacled by quantum fluctuations when approaching the Planck scale. In contrast, the idea of permutation merely presupposes the more fundamental principle of identity. This is why permutation groups may enter theoretical physics at finer levels of resolution and higher energies than the Lorentz group. Tetrons may be more basic than spinors[4].

[3]Actually, as discussed in section 1, T_d is not a subgroup of SO(3) because odd permutations correspond to certain roto-reflections in O(3). However, T_d is isomorphic to the octahedral group O which is a subgroup of SO(3).

[4]There is some remote similarity of this idea to a paper by Baugh et al [15].

As a consequence of the new representation a new type of particle statistics will arise (called *permutation* or *tetron statistics*) which differs from Fermi and Bose statistics and whose detailed properties will be discussed in the cause of this section.

However I want to postpone the presentation of the octonion Z_4 extension of S_4 to a later stage and at this point just *assume* that 4-tetron bound states have spin-$\frac{1}{2}$, in order to see, what consequences can be derived from this assumption.

First of all, the spin part of a 4-particle fermionic compound state must transform according to a projective representation of S_4, i.e. a representation of its covering group $\tilde{S}_4$. There are 3 irreducible projective representations of S_4, namely G_1, G_2 and H of dimensions 2, 2 and 4, respectively [6]. The sum 4+4+16 of the dimensions squared accounts for the 24 additional elements due to the Z_2 covering of S_4. G_1 uniquely corresponds to spin-$\frac{1}{2}$, i.e. is obtained as the restriction of the fundamental SU(2) representation to $\tilde{S}_4$. Similarly, H can be obtained from the spin-$\frac{3}{2}$ representation of SU(2), whereas G_2 is obtained from G_1 by reversing the sign for odd permutations and contains contributions from spins larger $\frac{3}{2}$.

For the understanding of the following arguments a short digression on quaternions and its usefulness for describing spin-$\frac{1}{2}$ fermions will be helpful:

Quaternions [9, 10, 11] are a non-commutative extension of the complex numbers and play a special role in mathematics, because they form one of only three finite-dimensional division algebra containing the real numbers as a subalgebra. (The other two are the complex numbers and the octonions.) As a vector space they are generated by 4 basis elementes 1, I, J and K which fulfill $I^2 = J^2 = K^2 = IJK = -1$, where K can be obtained as a product $K = IJ$ from I and J. Quaternions are non-commutative in the sense IJ=-JI.

Any quaternion q has an expansion of the form

$$\begin{aligned} q &= c_1 + Jc_2 \\ &= r_1 + Ir_2 + Jr_3 + Kr_4 \end{aligned} \tag{39}$$

with real r_i and complex c_i.

There is a one-to-one corresponence between unit quaternions q_0 and SU(2) matrices, because the latter are necessarily of the form $(\alpha, \beta; -\beta^*, \alpha^*)$ with complex α and β fulfilling $|\alpha|^2 + |\beta|^2 = 1$, and can be rewritten as $q_0 = \alpha + J\beta$.

SU(2) can be considered as a non-trivial Z_2-extension of SO(3), in the sense that to each element g of SO(3) 2 elements $\pm D(g)$ of SU(2) can be attributed. In fact any rotation g about an axis $n = (n_x, n_y, n_z)$ by an angle θ can be represented as

$$D(g) = \pm cos(\frac{\theta}{2}) \mp (n_x K + n_y J + n_z I) sin(\frac{\theta}{2}) \tag{40}$$

SU(2) matrices act on dublets of spinor fields (c_1, c_2) (c_1 with spin up and c_2 with spin down). In quaternion notation this action can be written as:

$$c_1 + Jc_2 \rightarrow (\alpha + J\beta)(c_1 + Jc_2) \tag{41}$$

For example the unit quaternions I and J corresponding to rotations by π about the x and y-axis amount to $c_1 \rightarrow Ic_1, c_2 \rightarrow -Ic_2$ and $c_1 \rightarrow -c_2, c_2 \rightarrow c_1$, respectively. For a general SU(2) transformation one has $c_1 \rightarrow \alpha c_1 - \beta^* c_2$ and $c_2 \rightarrow \alpha^* c_2 + \beta c_1$, from which e.g. the antisymmetric tensor product combination $c_1 c_2' - c_2 c_1'$ can be shown to be rotationally invariant (spin 0).

Coming back to the representation G_1, its symmetry function (also called G_1 in the following) can be written in terms of the G_1 representation matrices

(=unit quaternions) of $\tilde{S}_4$ as

$$\begin{aligned}
G_1 &= g(1,2,3,4) + Ug(2,3,1,4) + U^2 g(3,1,2,4) \\
&+ Ig(2,1,4,3) + Sg(3,2,4,1) + R^2 g(1,3,4,2) \\
&+ Jg(3,4,1,2) + Rg(1,4,2,3) + T^2 g(2,4,3,1) \\
&+ Kg(4,3,2,1) + Tg(4,1,3,2) + S^2 g(4,2,1,3) \\
&+ \frac{I+K}{\sqrt{2}} g(3,2,1,4) + \frac{I-J}{\sqrt{2}} g(1,3,2,4) + \frac{J+K}{\sqrt{2}} g(2,1,3,4) \\
&+ \frac{1-J}{\sqrt{2}} g(2,3,4,1) + \frac{1-K}{\sqrt{2}} g(3,1,4,2) + \frac{J-K}{\sqrt{2}} g(1,2,4,3) \\
&+ \frac{I-K}{\sqrt{2}} g(1,4,3,2) + \frac{1+K}{\sqrt{2}} g(2,4,1,3) + \frac{1+I}{\sqrt{2}} g(3,4,2,1) \\
&+ \frac{1+J}{\sqrt{2}} g(4,1,2,3) + \frac{I+J}{\sqrt{2}} g(4,2,3,1) + \frac{1-I}{\sqrt{2}} g(4,3,1,2) \qquad (42)
\end{aligned}$$

where $R = \frac{1}{2}(1-I-J-K), S = \frac{1}{2}(1-I+J+K), T = \frac{1}{2}(1+I-J+K)$ and $U = \frac{1}{2}(1+I+J-K)$. One can see explicitly from this equation, which S_4 permutation $\overline{ijkl}$ is represented in G_1 by which quaternion, because the corresponding quaternion appears as a coefficient of g(i,j,k,l). For example, the permutation $\overline{2341}$ is represented by $\pm(1-J)/\sqrt{2}$, and so on. In other words, the quaternion coefficients $1, I, J, K, (I+K)/\sqrt{2}, ..., R, S, T, ...$ in this equations represent the elements of $\tilde{S}_4$[5].

[5]While $\tilde{S}_4$ itself can be shown to make up the inner shell of D_4-lattices [16], the first half of coefficients in eq. (42) represent even permutations corresponding to $\tilde{A}_4$ which is sometimes called the 'binary tetrahedral group', and generates the F_4 lattice also called the ring of Hurwitz integers (=quaternions with half integer coefficients). The Hurwitz quaternions form a maximal order (in the sense of ring theory) in the division algebra of quaternions with rational components. This accounts for its importance. For example restricting to integer lattice points, which seems a more obvious candidate for the idea of an integral quaternion, one does not get a maximal order and is therefore less suited for developing a theory of left ideals as in algebraic number theory [3]. What Hurwitz realized, was that his definition of integral quaternions is the better one to operate with.

Due to the 2-fold covering of S_4 each of the real functions $g(i,j,k,l)$ in eq. (42) with its 24 terms is in fact a difference $p(i,j,k,l) - m(i,j,k,l)$ so as to obtain the 48 terms needed for a symmetry function of $\tilde{S}_4$.

$g(i,j,k,l)$ (or alternatively p and m) are the functions to be interpreted as tensor products of tetrons of the generic form

$$g(i,j,k,l) = a_i \otimes b'_j \otimes c''_k \otimes d'''_l \tag{43}$$

where a,b,c,d are the tetron 'flavor' and i,j,k,l their 'spin' indices. I.e. eq. (42) with (43) corresponds to the assumption that there are 4 tetron 'spins' $i,j,k,l \in \{1,2,3,4\}$ and that these add to give a spin-$\frac{1}{2}$ fermion.

It is true that the phenomenological observation of 24 quarks and leptons and their interactions imply a permutation principle only on the level of *inner* symmetries. However, the assumption of 4 different tetron spin within a fermion bound state comes closest to the original intuition of a spatial tetrahedral structure as discussed in section 1 and in ref. [2]. In case of e.g. only 2 tetron spins the G_1 function would look quite different, because in general, the permutation of the tensor product indices - denoted by primes in eq. (43) - must not be messed up with the permutation of spin states. Only in the case at hand, where 4 different spin states in 4 different tensor factors are considered, there is no difference.

Eq. (42) should be considered as the spin factor of the 4-tetron bound state (whereas the A_1, A_2, E, T_1 and T_2-functions implicitly given in tables 3 and 4 account for the flavor factor). In fact, working out the quaternion multiplications in eq. (42) and using $K = IJ$ one obtains a representation of the form $G_1 = c_1 + Jc_2$ with c_1 and c_2 decribing the 2 spin directions of the compound fermions, cf eq. (41). Mathematically, the appearance of 2 complex functions c_1 and c_2 in eq. (42) is merely expression of the fact that for the

2-dimensional representation G_1 4 real(=2 complex) symmetry functions can be constructed, which in eq. (42) are combined in one quaternion function.

The complete spin and flavor wave function of quarks and leptons can be written as

$$\begin{aligned} a_1 \otimes b'_2 \otimes c''_3 \otimes d'''_4 &+ b_1 \otimes c'_2 \otimes a''_3 \otimes d'''_4 + ... \\ I a_2 \otimes b'_1 \otimes c''_4 \otimes d'''_3 &+ ... \\ &... \end{aligned} \tag{44}$$

Here in the rows the tetron flavor indices a,b,c,d are permutated in order to obtain the appropriate flavor combination (A_1 of S_4 as an example), whereas in the columns the tetron spin indices 1,2,3,4 are permutated in order to obtain the G_1 spin combination.

On the microscopic level we shall be looking for a behavior of the spin indices 1,2,3,4, which mimic the behavior of G_1, i.e. which induce transformations of the tensor product combination $G_1 \rightarrow q_0 G_1$ for $q_0 \in \tilde{S}_4$, cf. eq. (41). E.g. one needs to show that $G_1 \rightarrow I G_1$ for rotations by π_x follows from a suitable transformation property of the tetrons.

Very important: eq. (42) reflects the statistical behavior of a 4-tetron-conglomerate unter permutations of its components. This behavior has a certain similarity to that of fermions but is certainly not identical. While conglomerates of fermions usually transform with the totally antisymmetric representation (like A_2), tetrons go with G_1, which gives a factor of I under the exchange $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ or $\frac{1}{\sqrt{2}}(J+K)$ under $(1 \leftrightarrow 2)$, whereas a 2-fermion conglomerate in a $A_2 = c_1 c'_2 - c_2 c'_1$ configuration responds with -1 (i.e. antisymmetric) to the exchange of $(1 \leftrightarrow 2)$. See table 6, where the behavior of tetrons and fermions is compared. The fact that tetrons behave more complicated under transpositions $(i \leftrightarrow j)$, has to do with the fact that

FERMIONS	TETRONS
behavior of the wave function under $2\pi n$ rotations:	
± 1 Z_2-extension	± 1 or $\pm L$ Z_4-extension
compound states:	
boson from 2 fermions: complex tensor product $A_2 = c_1 c_2' - c_2 c_1'$ bosonic behavior under rotations	fermion from 4 tetrons: quasi-complex, quaternion tensor product $G_1 = g(1,2,3,4) + Ig(2,1,4,3) + J...$ $= a_1 b_2' c_3'' d_4''' + I a_2 b_1' c_4'' d_3''' + ...$ fermionic behavior under rotations $G_1 \to (\alpha + J\beta) G_1$
permutation behavior/statistics:	
-1 under $(1 \leftrightarrow 2)$	a factor I under $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ a factor $\frac{1}{\sqrt{2}}(J+K)$ under $(1 \leftrightarrow 2)$ etc

Table 6: Comparison between the known fermion behavior and the anticipated tetron behavior.

transpositions in S_4 correspond to relatively complicated rotoreflections in T_d, c.f. table 1.

We therefore conclude that tetrons follow their own statistics which is neither bosonic nor fermionic, and conjecture, that a sort of 'tetron spin statistics theorem' holds, which will give an explanation of the selection rule proposed in section 4.

I now turn to the question, how such a behavior can be realized. The specific Z_4-extension of $\tilde{S}_4$ which I have in mind is constructed using octonions [9, 10, 11]. Octonions are a non-associative extension of the quaternions, i.e. they are a non-commutative non-associative division algebra spanned as a vector space by 8 elements 1, I, J, K, L, IL, JL and KL, i.e. any octonion t can be expanded in various ways like

$$t = q_1 + Lq_2 \tag{45}$$
$$= d_1 + Id_2 + Jd_3 + Kd_4 \tag{46}$$
$$= c_1 + Jc_2 + Lc_3 + JLc_4 \tag{47}$$
$$= r_1 + Ir_2 + Jr_3 + Kr_4 + Ls_5 + ILs_6 + JLs_7 + KLs_8 \tag{48}$$

with quaternions q_i, real numbers r_i and s_i, complex numbers c_i and $d_i = r_i + Ls_i$.

As an algebra the octonions can be generated from the trias I, J and L, because K, IL, JL and KL can be simply given as the product of I times J, I times L, J times L and K times L, respectively. They are non-associative in that $(IJ)L = -I(JL)$. In contrast to quaternions they cannot be representated as matrices.

Octonions are related to a number of exceptional structures in mathematics, among them the exceptional Lie groups [12], and have some applications in fields such as string theory and special relativity, where they are used to describe spinors in higher dimensional spacetimes, see e.g. [16].

Here I follow a different path and use them to construct a Z_4-extension of S_4, regarded as the subgroup T_d of spatial SO(3). This extension is defined as $\hat{S}_4 = \pm\tilde{S}_4(L)$, i.e. it consists of all elements q_0 of $\tilde{S}_4$ (as explicitly given in and after eq. (42)) and possibly multiplied with a factor of L. To be definite I always take multiplication from the right with brackets, i.e. $q_0(Lt)$, when acting on an arbitrary octonion t and denote this product by $q_0(L$. Using the octonion multiplication table [9, 10, 11] it can be shown that $\hat{S}_4$ is indeed

	R(π_x)=IL	R(π_y)=JL	R(π_z)=KL
R(π_x)=IL	-1	(KL)L	-(JL)L
R(π_y)=JL	-(KL)L	-1	(IL)L
R(π_z)=KL	(JL)L	-(IL)L	-1

Table 7: Behavior of the proposed octonion projective representation R for the SO(3) generators π_x, π_y and π_z. The appearance of octonion phases ± 1 and $\pm L$ is explicitly seen and proves that those objects do not give a true representation of the rotations but an octonion projective one.

closed under multiplication and thus offers a rather unique possibility to get a nontrivial Z_4-extension - in the same sense in which the octonions are a rather unique extension of the quaternion algebra.

The mentioned Z_4 is formed by the subset $\{\pm 1, \pm L\}$, and corresponds to a 4-fold ambiguity of the identity rotation. In general any element $\overline{ijkl} \in S_4$ is represented by 4 elements $\pm q_0$ amd $\pm q_0(L \in \hat{S}_4$. For example, a rotation by π around the x-axis, which in quaternion SU(2) notation is given by $\pm I$ has a 4-fold representation $\{\pm I, \pm I(L\}$ in $\hat{S}_4$.

I now use the following definition: **concerning spatial $S_4(= T_d)$ transformations a tetron is defined to be an octonion field t, on which $\hat{S}_4$ acts via octonion multiplication.**

In table 7, the action of this representation (called R) is given for the octonion generators IL, JL and KL. It could easily be extended to include any element $q_0(L \in \hat{S}_4$ and proves the appearance of phases $p(g_1, g_2) \in \{\pm 1, \pm L\}$ in the products $R(g_1)R(g_2) = p(g_1, g_2)R(g_1 g_2)$ for $g_1, g_2 \in \hat{S}_4$.

Transformations of the form $q_0(L$ are conjectured to give fermionic behavior when acting on a 4-fold tensor product of tetrons. The point is that loosely speaking applying $q_0(L$ four times one ends up with a factor $q_0 L^4 = q_0$, i.e. fermion behavior.

Unfortunately the definition of a tensor product is not unique. A priori one could use either a
i) a real tensor product,
ii) a quaternion tensor product with respect to any quaternion subalgebra of the octonions or
iii) a complex tensor product with respect to any embedding of the complex numbers.

At the moment of writing I am still analyzing the various possibilities. For example, one may define a tensor product of 4 tetron fields t, t', t'' and t''' as

$$g(i,j,k,l) \;=\; d_i \otimes d'_j \otimes d''_k \otimes d'''_l + o.c. \tag{49}$$

where d_i are defined in eq. (46) and o.c. denotes the octonion conjugate. The $g(i,j,k,l)$ so defined should be inserted in the fermion symmetry function G_1, eq. (42).

Eq. (49) corresponds to case iii. Another good choice might be a tensor product of angular momentum eigenstates. There is then an ambiguity in whether I or IL should be chosen as the angular momentum operator. Furthermore, one may use either the complex eigenstates c_i of eq. (47) (case iii) or quaternion eigenstates which are typically of the form $c_1 + J(Lc_4)$ etc (case ii).

In any of the cases one should work out the transformation behavior induced on G_1 by transformations of the components. For example, for the choice eq. (49) one has:

- Under an identity transformation L a tetron t transforms as

$$d_1 + Id_2 + Jd_3 + Kd_4 \rightarrow d_1 + I(-Ld_2) + J(-Ld_3) + K(-Ld_4) \quad (50)$$

  and therefore $g(i, j, k, l) \rightarrow -g(i, j, k, l)$ and $G_1 \rightarrow -G_1$, which is exactly the behavior of a fermion under the identity rotation.

- Under a transformation $R(\pi_x) = I(L$ the tetron transforms as $I(Lt) = -d_2 + I(Ld_1^*) + J(-Ld_4^*) + K(Ld_3^*)$ which gives $g(1, 2, 3, 4) \rightarrow -g(2, 1, 4, 3)$, whereas $g(2, 1, 4, 3) \rightarrow +g(1, 2, 3, 4)$ etc, which leads to $G_1 \rightarrow IG_1$ as needed.

and this analysis should be continued scanning all elements of S_4 and all possible tensor products whether they yield the required behavior. The results will be presented in a future publication [4].

7 Summary

It is a remarkable observation, that quarks, leptons and gauge bosons can be ordered with the help of essentially the same symmetry group, the permutation group S_4.

Starting from this paradigma we have seen, that and how from the 24x24=512 possible fermion-antifermion product states 12 are selected to describe the gauge bosons, and - though lacking an understanding of the underlying dynamics responsible for this selection - by inspection of Clebsch-Gordon coefficients we have tried to follow the path of how this dynamics unfolds itself on the level of gauge bosons.

Realizing that there might be a connection of the S_4-states to representations of SU(4) we have found two options which match the phenomenological fermion and gauge boson spectrum equally well: either one uses a continuous inner symmetry group like SU(4) together with an exclusion principle or one sticks to the discrete permutation symmetry.

The discussion of SU(4) suggests the existence of a fundamental quartet of 'tetron' constituents. Taking that idea serious one encounters the difficulty to generate the spin-$\frac{1}{2}$ behavior of quarks and leptons from 4 constituents by conventional means.

In section 6 a suggestion for the spatial transformation properties of tetrons was made which relies on an octonion Z_4-extension of the spatial S_4 permutation symmetry. It is certainly true that the phenomenological observation of 24 quarks and leptons and their interactions suggest a permutation principle only on the level of *inner* symmetries. However due to the problems which arise in connection with spin and statitistics one is naturally lead to consider the possibility that inner and outer permutation behavior may be intertwined and that this can be used to understand the spin-$\frac{1}{2}$ nature of quarks and leptons.

If this approach has some meaning it is possible that besides G_1 also the two other half-integer spin representations of $\tilde{S}_4$ (H and G_2) play a role in nature, or in other words, that particles with spin $\frac{3}{2}$ and $\frac{5}{2}$ may appear at higher energy levels.

Another speculation concerns the inclusion of gravity in the tetron framework, i.e. the question whether the underlying unknown interaction of tetrons may also be used to describe gravitons.

Acknowledgement

Table 3 is due to Pete Taylor from the Theoretical Chemistry Department

of Warwick University. I owe him many fruitful e-mail discussions.

References

[1] R.N. Mohapatra and J.C. Pati, Phys. Rev. D11 (1975) 566, 2558.

[2] B. Lampe, J. Phys. G 34 (2007) 1 and arXiv:hep-ph/0610270 (2006); see also arXiv:hep-ph/9810417 (1998)

[3] B.L. van der Waerden, Algebra, Springer Verlag.

[4] B. Lampe, in preparation.

[5] P.R. Taylor, Molecular Symmetry and Quantum Chemistry, in Lecture Notes in Quantum Chemistry, ed. B. Roos (Springer Verlag, Berlin 1992).

[6] R.C. Johnson, Phys. Lett. 114B (1982) 147.

[7] M.E. Peskin and T. Takeuchi, Phys. Rev. Lett. 65, 964 (1990).

[8] J.S. Griffith, Irreducible Tensor Methods for Molecular Symmetry Groups, Dover Publications, 2006.

[9] J. Conway and D. Smith, On Octonions and Quaternions, Peters Publishing, ISBN 1-56881-134-9, Natick, MA (2003).

[10] I. L. Kantor and A. S. Solodovnikov, Hypercomplex Numbers - an Elementary Introduction to Algebras, Springer Verlag, Berlin, 1989.

[11] R.D. Schafer, An Introduction to Nonassociative Algebras, Academic Press, New York and London, 1966.

[12] A. Sudbery, J. Math. Phys. 24 (1983), 1986.

[13] P. Jordan, J. von Neumann and E.P. Wigner, Ann. Math. N.Y. 35, 29 (1934).

[14] S. Adler, Quaternion Quantum Mechanics and Quantum Fields, Oxford UP, NewYork.

[15] J. Baugh, D. Finkelstein, A. Galiautdinov, and H. Saller, J. Math. Phys. 42, 1489 (2001).

[16] G.M. Dixon, Invariant Representations of the Leech Lattice, hep-th9504040, 1995, and references therein.